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# THESIS

## DYNAMIC STABILITY OF FLIGHT VEHICLES

by

Dimitrios Panagiotis Pouliezos

June 1982

Thesis Advisor: D. M. Layton

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This thesis presents an analytical treatment of the dynamics of the flight vehicle and might be used as a textbook for a Dynamic Stability and Control advanced class.

Concentration is given to derivation of equations of motion, investigation of particular modes of motion, stability derivatives, aerodynamic transfer functions and digital computer solutions.

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# Dynamic Stability of Flight Vehicles

by

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Lieutenant, Greek Navy
B.S.E.E., Naval Postgraduate School, 1981

Submitted in partial fulfillment of the requirements
for the degree of

MASTER OF SCIENCE IN ENGINEERING SCIENCE

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Concentration is given to derivation of equations of motion, investigation of particular modes of motion, stability derivatives, aerodynamic transfer functions and digital computer solutions.

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# PREFACE

The Department of Aeronautics, Naval Postgraduate School offers several courses at different levels on the dynamics of flight vehicles. These courses include studies of fixed wing stability, rotary wing stability, and the stability of missiles. In addition, there are general courses that cover the dynamics of bodies in six degrees of freedom, including the development of the equations of motion.

There is currently no signle resource available that covers the material for the airplane and missile courses, as well as the introductory dynamics classes, and, as a result, the instructors must make use of large numbers of locally prepared handouts and/or assign several textbooks, only portions of which will be used.

In order to provide a solution to this problem, the following material has been developed for use in the dynamics classes in the Department of Aeronautics. Although this material is, in places, of greater depth than what would be covered in the courses, it is felt that the value of this material is enhanced by its degree of thoroughness.

## CHAPTER 1

## DYNAMICS OF MOTION

### 1.1 INTRODUCTION

This chapter is intended to be a formal review of mathematical and physical concepts that will be used to develop the theory of dynamic stability of flight vehicles.

The starting point is the basic laws of Newtonian Mechanics, with a useful application to spring-mass-damper systems. We shall find later how the motion of any conventional flight vehicle is similar to those systems.

We then proceed to rigid body equations of motion, described in non-inertial reference frames.

### 1.2 NEWTON'S LAWS OF MOTION

In any system of units, basic quantities and units are related through Newton's laws of motion.

NEWTON'S 1ST LAW OR THE LAW OF COORDINATE SYSTEMS:

If there is no force acting on a particle, its velocity remains constant.

This law is often called the law of inertia and applies to inertial frames of reference.

NEWTON'S 2ND LAW OR THE LAW OF DYNAMICS:

The time rate of change of the linear momentum of a particle, is proportional to the net force acting on it.

The vector quantity

$$\vec{p}(t) = m(t) \cdot \vec{v}(t) \tag{1-1}$$

is defined as the linear momentum of the particle, where m(t) is the inertial mass of the particle and v(t) is its velocity relative to the reference frame in question.

The 2nd law is mathematically expressed as

$$\vec{F}_{net} = \frac{1}{g_c} \cdot \frac{d\vec{p}(t)}{dt} \tag{1-2}$$

or

$$\vec{F}_{net} = \frac{1}{g_c} \cdot \frac{d}{dt} \left[ m(t) \cdot \vec{v}(t) \right] \tag{1-3}$$

If the mass m remains constant, we can write:

$$\vec{F}_{net} = \frac{1}{g_c} m \frac{d}{dt} \vec{v}(t) \tag{1-4}$$

or

$$\vec{F}_{net} = \frac{1}{g_c} m \vec{a}(t) \tag{1-5}$$

where $\vec{a}(t)$ is the acceleration caused by the net applied force $\vec{F}_{net}$.

The proportionality factor is expressed as $1/g_c$ and its value depends on the system of units being used.

In the English Engineering system, one pound force (1-lbf) will cause to one pound mass (1-lbm), an acceleration of 32.174 ft/sec$^2$. Thus substituting into Equation 1-5

$$1\text{-lbf} = \frac{1}{g_c} (\text{lbm}) \cdot 32.174 \ (\text{ft/sec}^2) \tag{1-6}$$

Hence

$$g_c = 32.174 \quad \mathrm{lbm\ ft/lbf \cdot sec^2} \tag{1-7}$$

A few other units used for velocity, are given below with their corresponding conversion factors:

1 mile per hour (1mph) = 1.4666 ft/sec

1 knot = 1 nautical mile per hour = 1.6878 ft/sec

NEWTON'S 3RD LAW OR THE LAW OF MUTUAL FORCES:

If one body exerts a force on a second body, the latter exerts an equal but opposite force on the first.

This law allows us to define the gravity force, or the weight of a body, that has a given mass (m). It is the force that results due to the earth's attraction.

Besides the three fundamental laws of mechanics, the Newton's Universal Gravitational Law states that two inertial masses m1 and m2, separated by a distance R are subject to an attraction force according to the equation:

$$F = g_c G \frac{m_1 m_2}{R^2} \tag{1-8}$$

where G, is the gravitational constant, which in the EE system is $G = 6.32 \times 10^{-11} \ \mathrm{lbf\text{-}ft^2/lbm^2}$

If now $m_1 = M_e = 1.32 \times 10^{25}$ lbm (mass of the earth), then near earth's surface where $R = R_e = 2.09 \times 10^{7}$ ft (radius of the earth), we can write:

$$W = g_c \frac{G M_e}{R_e^2} \cdot m \tag{1-9}$$

or

$$W = mg \tag{1-10}$$

where $g = g_c G M_e / R_e^2 = 32.174$ ft/sec$^2$, called acceleration of gravity.

By Equation 1-10 we can see that 1-lbm, subject to the earth's gravitational field, experiences a force 1-lbf and hence it accelerates with 32.174 ft/sec$^2$, near earth's surface.

We can also define another unit of mass, the slug, as the mass that experiences an acceleration 1-ft/sec$^2$, if subject to a force of 1-lbf.

By this definition

$$1 \text{ slug} = 32.174 \text{ lbm} \tag{1-11}$$

Consequently, we can derive the density ($\rho$), as the mass per unit volume usually measured in slugs/ft.

$$\text{density } (\rho) = \frac{\text{mass } (m)}{\text{volume } (v)} \tag{1-12}$$

People in the field use the (g), as a short hand notation for force or acceleration. A 5-g force simply means a force 5 times the body's weight.

## 1.3 ANGULAR MOMENTUM

A special significance to the 2nd law is given by the angular momentum approach derived simply by crossing the radius vector $\vec{r}$, on to the 2nd law.

$$\vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F}_{net} \tag{1-13}$$

We define

$$\vec{G}_{net} = \vec{r} \times \vec{F}_{net} \tag{1-14}$$

the net torque or moment of the force about the origin and

$$\vec{h} = \vec{r} \times \vec{p} \tag{1-15}$$

the angular momentum about the origin.

Differentiating Equation 1-15 with respect to time, we get

$$\frac{d\vec{h}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \tag{1-16}$$

The first term is zero, because

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times (m\vec{v}) = m(\vec{v} \times \vec{v}) = 0 \tag{1-17}$$

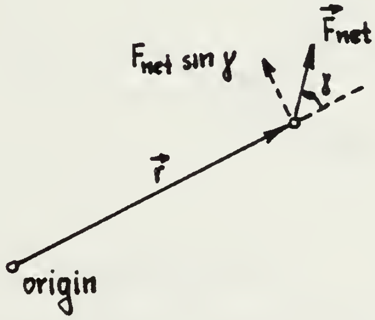


Figure 1-1

Moment of Force About the Origin

So we finally get

$$\frac{d\vec{h}}{dt} = \vec{G}_{net} \tag{1-18}$$

which tells us that the time rate of change of the angular momentum of a particle, equals to the net torque or moment acting on it.

If no torque is acting on a particle, its angular momentum is conserved, likewise its linear momentum is conserved, if no force is acting on it.

A simple example

For a pendulum that swings in a vertical plane, having a mass (m), the acting forces are, its weight W = mg and the tension of the rod T. The rod is assumed to have negligible mass.

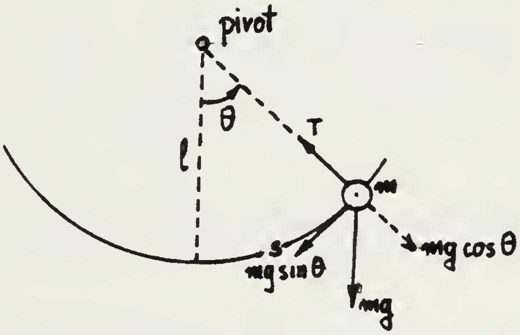


Figure 1-2

A Simple Pendulum

If θ is the angle between the rod and the vertical axis, s, is the arc length corresponding to the angle θ and l, is the length of the rod, a direct application to the 2nd law (Equation 1-5), gives:

$$-mg\sin\theta = m\ddot{s} \tag{1-19}$$

and equilibrium condition in the direction of the rod, gives:

$$-mg\cos\theta = T \tag{1-20}$$

Equation 1-19, can be written

$$\ddot{s} + g\sin\theta = 0 \tag{1-21}$$

If $\theta$ is small, we can assume

$$\theta = \frac{s}{\ell} \quad \Rightarrow \quad s = \ell\theta \tag{1-22}$$

and

$$\theta \cong \sin\theta \tag{1-23}$$

With this assumption, Equation 1-21, can be written

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0 \tag{1-24}$$

Equation 1-24, is a linear differential equation, with constant coefficients, which describes the motion of the pendulum.

The same result can be derived, if we use the angular momentum approach.

The angular momentum of the pendulum, about the pivot, is simply ($\ell$), cross the linear momentum ($m\dot{s}$). The time rate of change of the angular momentum, has to be equal to the net external torque.

$$\frac{d}{dt}(\ell \times m\dot{s}) = -\ell mg\sin\theta \tag{1-25}$$

Since $\ell$ is constant ($d\ell/dt = 0$), we can write

$$\ell m\ddot{s} = -\ell mg\sin\theta \tag{1-26}$$

or

$$\ddot{s} + mg\sin\theta = 0 \tag{1-27}$$

or

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0 \tag{1-28}$$

The minus sign in Equations 1-19 and 1-25, indicates that if the angle $\theta$ decreases, the acceleration $\ddot{s}$ in Equation 1-19 and the angular momentum in Equation 1-25, increase.

## 1.4 UNDAMPED HARMONIC OSCILLATOR

In this section and in the following section, we will derive the equations and the solutions to the spring mass damper systems.

The undamped oscillator is simply a spring mass system without friction, shown in Figure 1-3.

Let k, denote the spring stiffness, m, the mass of the body, x(t), the displacement of the body from the equilibrium position. Notice that the weight of the body stretches the spring permanently, so that $x_e$ is the equilibrium position, after the body has hanged to the spring.

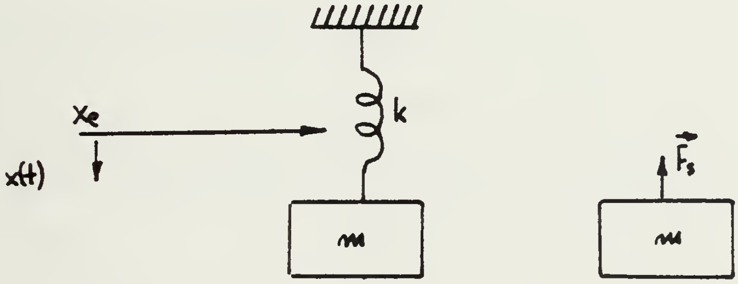


Figure 1-3

Spring Mass System

The restoring force according to Hook's law, is

$$\bar{F}_s = -kx \tag{1-29}$$

Then according to the 2nd law

$$m\ddot{x} = -kx \tag{1-30}$$

or

$$\ddot{x} + \frac{k}{m}x = 0 \tag{1-31}$$

As an example we will take

$$m = 18 \text{ lbm} \tag{1-32}$$

$$k = 40.5 \text{ lbf/ft} \tag{1-33}$$

If we let

$$\omega_n^2 = g_c \frac{k}{m} \tag{1-34}$$

where $\omega_n$ is the so called natural frequency of oscillations, Equation 1-31 becomes

$$\ddot{x} + \omega_n^2 x = 0 \tag{1-35}$$

This is a linear differential equation with constant coefficients. Its characteristic equation is clearly

$$\lambda^2 + \omega_n^2 = 0 \tag{1-36}$$

which has characteristic roots

$$\lambda = \pm j\omega_n \tag{1-37}$$

The most general solution of Equation 1-35 can be expressed as

$$x(t) = (A_+)e^{j\omega_n t} + (A_-)e^{-j\omega_n t} \tag{1-38}$$

where $A_+$ and $A_-$ are constants depending on the initial conditions.

Expanding the exponential terms of Equation 1-38, we obtain

$$x(t) = \alpha \cos \omega_n t + \beta \sin \omega_n t \tag{1-39}$$

where

$$\alpha = A_+ + A_- \tag{1-40}$$

and

$$\beta = j(A_+ - A_-) \tag{1-41}$$

If we now express

$$\sin \omega_n t = \cos(\omega_n t + 90°) \tag{1-42}$$

we can give the following useful form to our solution

$$x(t) = x_{max} \cos(\omega_n t + \theta_o) \tag{1-43}$$

where $x_{max}$ is the maximum deviation from the equilibrium position and $\theta_o$, the initial phase.

We can relate the quantities $x_{max}$ and $\theta_o$, with the initial conditions, as follows:

At time t=0, the initial displacement $x_o$, is given by Equation 1-43.

$$x_o = x_{max} \cos \theta_o \tag{1-44}$$

and the initial velocity $\dot{x}_o$, is given by differentiating Equation 1-43 with respect to time and evaluating at t=0.

$$\dot{x}_o = -x_{max}\, \omega_n \sin \theta_o \tag{1-45}$$

As an example we will take

$$x_o = 4 \text{ ft} \tag{1-46}$$

$$\dot{x}_o = 32 \text{ ft/sec} \tag{1-47}$$

From Equations 1-44 and 1-45, the initial phase and the maximum displacement can be found to be

$$\theta_o = \tan^{-1}\left(\frac{\dot{x}_o}{-\omega_n x_o}\right) \tag{1-48}$$

$$x_{max} = \pm\left(x_o^2 + \frac{\dot{x}_o^2}{\omega_n^2}\right)^{1/2} \tag{1-49}$$

Note: In Equation 1-48, keep the sign, so that the quantity in brackets will give the right quadrant for the angle $\theta_o$.

For our example

$$\omega_n^2 = g_c \frac{K}{m} = 32.174 \times \frac{40.5}{18} = 79.39 \text{ sec}^{-2} \tag{1-50}$$

$$\theta_o = \tan^{-1}\left(\frac{32}{-8.51 \times 4}\right) = 136.76^\circ = 2.39 \text{ rad} \tag{1-51}$$

$$x_{max} = \pm\left(16 + \frac{1024}{72.39}\right)^{1/2} = \pm 5.49 \text{ ft} \tag{1-52}$$

so that the response is

$$x(t) = -5.49 \cos(8.51\, t + 2.39 \text{ rad}) \tag{1-53}$$

## 1.5 DAMPED HARMONIC OSCILLATOR

Consider the spring mass damper system shown in Figure 1-4. Assume that the friction force exerted by the damper, is a linear function of the mass velocity.

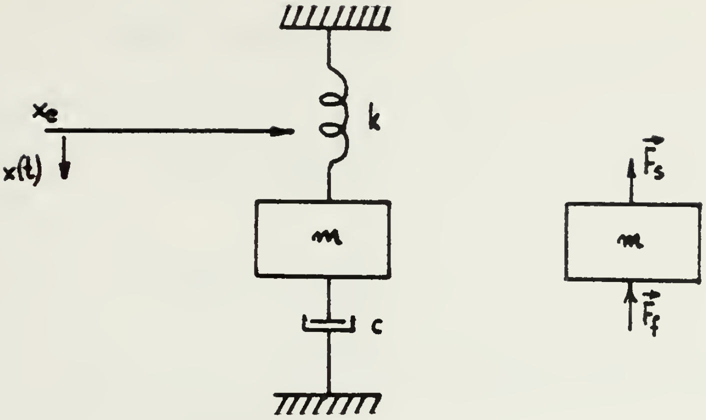


Figure 1-4

Spring Mass Damper System

Let k the spring stiffness, m the mass of the body, x(t) the displacement of the body from the equilibrium position $x_e$ , as it was defined in section (1.4), and c, the coefficient of friction.

The restoring force is again

$$F_s = -kx \tag{1-54}$$

and the friction force is

$$F_f = -c\dot{x} \tag{1-55}$$

Then according to the 2nd law

$$m\ddot{x} = -c\dot{x} - kx \tag{1-56}$$

or

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{1-57}$$

As an example we will take

$$m = 18 \text{ lbm} \tag{1-58}$$

$$k = 40.5 \text{ lbf/ft} \tag{1-59}$$

If we let

$$\omega_n^2 = g_c \frac{k}{m} \tag{1-60}$$

and

$$2\zeta\omega_n = g_c \frac{c}{m} \tag{1-61}$$

where $\zeta$ is the so called damping coefficient, then the Equation 1-57 is written

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0 \tag{1-62}$$

This is a linear differential equation with constant coefficients. Its characteristic equation is clearly

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \tag{1-63}$$

Three cases arise in obtaining solutions to the differential Equation 1-62.

As an example, we will take

$$x_0 = 4 \text{ ft} \tag{1-64}$$

$$\dot{x}_0 = 32 \text{ ft/sec} \tag{1-65}$$

Case 1: Critically Damped Motion ($\zeta = 1$)

As an example we will take

$$c = 9.52 \text{ lbf/ft-sec}^{-1} \tag{1-66}$$

so that

$$\zeta = g_c \frac{c}{2m\omega_n} = 1 \tag{1-67}$$

In this case, the characteristic equation has a double root

$$\lambda = -\omega_n \tag{1-68}$$

and the most general solution is

$$x(t) = (A_+ + A_- t)\, e^{-\omega_n t} \tag{1-69}$$

where $A_+$ and $A_-$ are constants determined by the initial conditions as follows

$$A_+ = x_0 \tag{1-70}$$

and

$$A_- = \dot{x}_0 + x_0 \omega_n \tag{1-71}$$

so that the solution in terms of the initial conditions can be written

$$x(t) = \left[x_0 (1 + \omega_n t) + \dot{x}_0 t\right] e^{-\omega_n t} \tag{1-72}$$

For our example

$$x(t) = \left[4 (1 + 8.51 t) + 32 t\right] e^{-8.51 t} \tag{1-73}$$

or

$$x(t) = (4 + 66.04 t)\, e^{-8.51 t} \tag{1-74}$$

Case 2: Overdamped Motion ($\zeta > 1$)

As an example we will take

$$c = 14.28 \text{ lbf/ft-sec}^{-1} \tag{1-75}$$

so that

$$\zeta = g_c \frac{c}{2m\omega_n} = 1.5 \tag{1-76}$$

In this case the roots of the characteristic equation are real and negative

$$s = -\zeta\omega_n \pm \omega_n (\zeta^2 - 1)^{1/2} \tag{1-77}$$

and the most general solution is

$$x(t) = e^{-\zeta\omega_n t}\left[A_+ e^{\omega_n\sqrt{\zeta^2-1}\, t} + A_- e^{-\omega_n\sqrt{\zeta^2-1}\, t}\right] \tag{1-78}$$

where $A_+$ and $A_-$ are determined by the initial conditions as follows

$$x_0 = A_+ + A_- \tag{1-79}$$

$$\dot{x}_0 = -\zeta\omega_n (A_+ + A_-) + \omega_n \sqrt{\zeta^2-1}\,(A_+ - A_-) \tag{1-80}$$

which yields

$$A_+ = \frac{(\sqrt{\zeta^2-1}+\zeta)\,x_0 + \dfrac{\dot{x}_0}{\omega_n}}{2\sqrt{\zeta^2-1}} \tag{1-81}$$

$$A_- = \frac{(\sqrt{\zeta^2-1}-\zeta)\,x_0 - \dfrac{\dot{x}_0}{\omega_n}}{2\sqrt{\zeta^2-1}} \tag{1-82}$$

For our example

$$A_+ = 6.36 \text{ ft} \tag{1-83}$$

$$A_- = -2.36 \text{ ft} \tag{1-84}$$

and

$$x(t) = e^{-12.77t}\left(6.36\, e^{9.51t} - 2.36\, e^{-9.51t}\right) \tag{1-85}$$

or

$$x(t) = 6.36\, e^{-3.26t} - 2.36\, e^{-22.28t} \tag{1-86}$$

### Case 3: Underdamped Motion ($\zeta<1$)

As an example we will take

$$c = 4.76 \text{ lbf/ft-sec}^{-1} \tag{1-87}$$

so that

$$\zeta = g_c \frac{c}{2m\omega_n} = 0.5 \tag{1-88}$$

In this case the roots of the characteristic equation are imaginary with negative real parts.

$$s = -\zeta\omega_n \pm j\,\omega_n \sqrt{1-\zeta^2} \tag{1-89}$$

If we define

$$\omega_d = \omega_n \sqrt{1-\zeta^2} \qquad (\text{for } \zeta<1) \tag{1-90}$$

the so called damped frequency of oscillations,

$$s = -\zeta\omega_n \pm j\,\omega_d \tag{1-91}$$

and the most general solution is

$$x(t) = e^{-\zeta\omega_n t}\left(A_+ e^{j\omega_d t} + A_- e^{-j\omega_d t}\right) \tag{1-92}$$

Expanding the exponential terms we obtain

$$x(t) = e^{-\zeta\omega_n t}\left(\alpha \cos\omega_d t + \beta \sin\omega_d t\right) \tag{1-93}$$

where

$$\alpha = A_+ + A_- \tag{1-94}$$

and

$$\beta = j\,(A_+ - A_-) \tag{1-95}$$

The solution can be formulated as

$$x(t) = X\, e^{-\zeta\omega_n t} \cos(\omega_d t + \Theta_o) \tag{1-96}$$

where $X$ and $\Theta_o$ are constants determined by the initial conditions as follows: From Equation 1-69 we obtain

$$e^{\zeta\omega_n t}\, x(t) = X \cos(\omega_d t + \Theta_o) \tag{1-97}$$

Differentiating Equation 1-97 with respect to time we obtain

$$e^{\zeta\omega_n t}\left[\zeta\omega_n\, x(t) + \dot{x}(t)\right] = -X\,\omega_d \sin(\omega_d t + \Theta_o) \tag{1-98}$$

For t=0, Equations 1-97 and 1-98 become respectively

$$x_o = X \cos\Theta_o \tag{1-99}$$

and

$$\zeta\omega_n x_o + \dot{x}_o = -X\omega_d \sin\theta_o \qquad (1\text{-}100)$$

So the constants are determined as

$$\theta_o = \tan^{-1}\left[\zeta\omega_n x_o + \frac{\dot{x}_o}{-\omega_d x_o}\right] \qquad (1\text{-}101)$$

and

$$X = \pm\left[x_o^2 + \frac{(\zeta\omega_n x_o + \dot{x}_o)^2}{\omega_d^2}\right]^{1/2} \qquad (1\text{-}102)$$

Note: In Equation 1-101, keep the sign, so that the quantity in brackets will give the right quadrant for the angle $\theta_o$.

For our example

$$\omega_d = 8.51\,(1-0.5^2)^{1/2} = 7.37 \text{ sec}^{-1} \qquad (1\text{-}103)$$

$$\theta_o = 121.02^\circ = 2.11 \text{ rad} \qquad (1\text{-}104)$$

$$X = \pm\, 7.76 \text{ ft} \qquad (1\text{-}105)$$

so that the response is

$$x(t) = -7.76\, e^{-4.26t} \cos(7.37t + 2.11 \text{ rad}) \qquad (1\text{-}106)$$

The roots of the characteristic equation can be plotted in the complex plane as shown in Figure 1-5.

In cases 1 and 2, the damping is so large that no oscillations take place and the mass simply returns gradually to the equilibrium position x=0. In those cases the roots are real and negative.

In case 3, the damping has been reduced, so that oscillations about the equilibrium position, do take place, the amplitude of which dies out as $t\to\infty$.

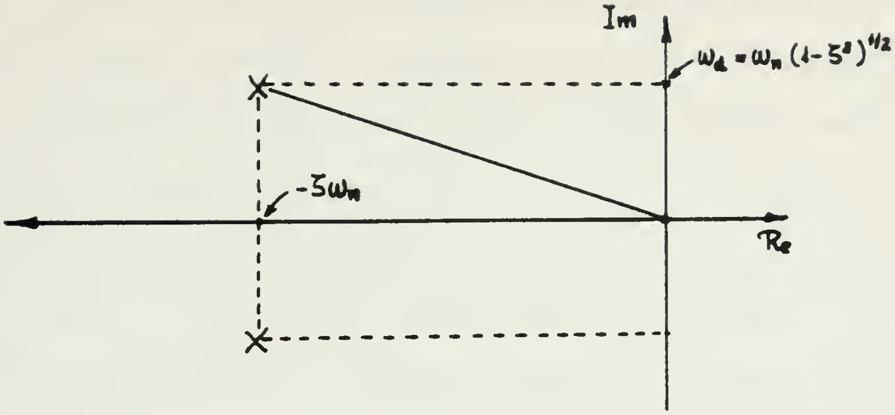


Figure 1-5

Plot of the Characteristic Roots in the Complex Plane

Figure 1-6 shows the responses in the above three described cases where it is assumed that the initial displacement is $x_o$ , and the initial velocity is $\dot{x}_o = 0$.

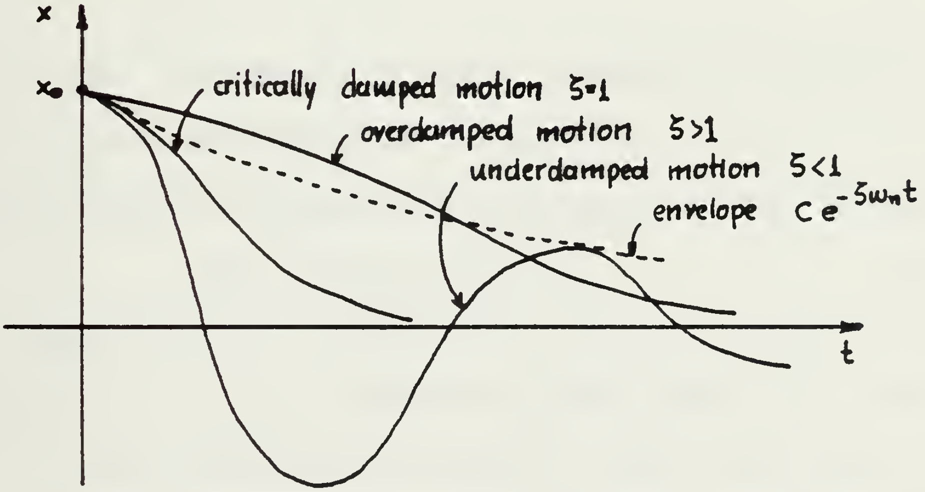


Figure 1-6

Response of Displacement (x) of a Spring Mass Damper System

Notice that in the case where the damping $\zeta$ is zero, we have a spring mass system, without friction, described in section 1.4. In this case, the roots are imaginary and $\omega_d$ is the same as $\omega_n$ (Figure 1-5).

It is clear that the system's physical constants (in our case $\zeta$,m,c,k), uniquely define the roots of the characteristic equation. Those roots are given the name: normal modes because one can determine from those roots, the corresponding characteristic motion (in our case, undamped critically damped, overdamped, underdamped).

We have seen so far how an harmonic damped oscillator responds, if initially disturbed from its equillibrium position. Then knowing the initial conditions we can always find the response of the displacement or velocity versus time.

Now we are interested in what happens if the oscillator is driven by an external input force F(t).

In this case the solution will consist of two parts:

$$x(t) = x_{hom}(t) + x_{part}(t) \qquad (1\text{-}107)$$

where $x_{hom}(t)$ is the homogenous part of the solution discussed so far, i.e. considering the input force zero and $x_{part}(t)$ is the particular solution due to this particular applied force, i.e. considering the initial conditions zero.

To obtain particular solutions one can use the method of undetermined coefficients, the method of variation of parameters or the Laplace Transform method.

In illustrating the above methods we are considering the following cases:

(a) Sinusoidal Inputs

The spring mass damper is driven bythe sinusoidal force

$$F(t) = F_0 \cos \omega t \tag{1-108}$$

As an example, we will take

$$F(t) = 24 \cos 3.5 t \tag{1-109}$$

The method of undetermined coefficients assumes a superposition of sinusoidal solutions of the form

$$x_{part}(t) = A_+ \cos \omega t + A_- \sin \omega t \tag{1-110}$$

or equivalently

$$x_{part}(t) = X \cos(\omega t + \theta_0) \tag{1-111}$$

where $X$ and $\theta_0$ are the coeffieients to be evaluated.

Substituting into the differential equation

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = F_0 \cos \omega t \tag{1-112}$$

we obtain

$$(\omega_n^2 - \omega^2)\cos(\omega t + \theta_0) - 2\zeta\omega_n\omega \sin(\omega t + \theta_0) = \frac{F_0}{X} \cos \omega t \tag{1-113}$$

Expanding cosine and sine and collecting terms, we get

$$\begin{aligned}&\left[(\omega_n^2 - \omega^2)\cos\theta_0 - 2\zeta\omega_n\omega \sin\theta_0\right]\cos\omega t \\ &\quad -\left[(\omega_n^2 - \omega^2)\sin\theta_0 + 2\zeta\omega_n\omega\cos\theta_0\right]\sin\omega t = \frac{F_0}{X}\cos\omega t\end{aligned} \tag{1-114}$$

Equating coefficients

$$(\omega_n^2 - \omega^2)\cos\theta_0 - 2\zeta\omega_n\omega\sin\theta_0 = \frac{F_0}{X} \tag{1-115}$$

and

$$(\omega_n^2 - \omega^2)\sin\theta_0 + 2\zeta\omega_n\omega\cos\theta_0 = 0 \tag{1-116}$$

From Equation 1-116 we obtain

$$\tan\theta_o = \frac{-2\zeta\omega_n\omega}{\omega_n^2-\omega^2} \qquad (1\text{-}117)$$

Multiplying Equation 1-115 by $\cos\theta_o$, Equation 1-116 by $\sin\theta_o$, and by adding, we obtain

$$(\omega_n^2-\omega^2) = \frac{F_o}{X}\cos\theta_o \qquad (1\text{-}118)$$

The $\cos\theta_o$ is expressed in terms of $\tan\theta_o$ as

$$\cos\theta_o = \frac{1/\tan\theta_o}{\left[1+\left(\frac{1}{\tan\theta_o}\right)^2\right]^{1/2}} = \frac{\omega_n^2-\omega^2}{\left[(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2\right]^{1/2}} \qquad (1\text{-}119)$$

so that

$$X = \frac{F_o}{\left[(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2\right]^{1/2}} \qquad (1\text{-}120)$$

and

$$\theta_o = \tan^{-1}\frac{-2\zeta\omega_n\omega}{\omega_n^2-\omega^2} \qquad (1\text{-}121)$$

For our example with the three different values of $\zeta$ considered, the responses are:

for $\zeta=1$ $\quad x_{part}(t) = 0.28\cos(3.5t+5.5\text{ rad})$ (1-122)

for $\zeta=1.5$ $\quad x_{part}(t) = 0.22\cos(3.5t+5.31\text{ rad})$ (1-123)

for $\zeta=0.5$ $\quad x_{part}(t) = 0.36\cos(3.5t+5.82\text{ rad})$ (1-124)

The total responses correspondingly are:

$$x(t) = (4+66.04t)e^{-8.51t} + 0.28\cos(3.5t+5.5\text{ rad}) \qquad (1\text{-}125)$$

$$x(t) = 6.36e^{-3.26t} - 2.36e^{-22.28t} + 0.22\cos(3.5t+5.31\text{ rad}) \qquad (1\text{-}126)$$

$$x(t) = -7.76\, e^{-4.26t} \cos(7.37t + 2.11 \text{ rad}) + 0.36 \cos(3.5t + 5.82 \text{ rad}) \quad (1\text{-}127)$$

(b) Singularity Inputs

The Laplace Transform method can be very easily applied, since we are dealing with linear differential equations having constant coefficients.

Appendix A gives a short Laplace Transform Table and some Laplace Transform Theorems that help to transform differential equations to algebraic.

In illustrating the above method, we are considering unit impulse and unit step input forces.

(1) Unit Impulse Response

The spring mass damper is driven by the singularity input

$$F(t) = \delta(t) \quad (1\text{-}128)$$

The differential equation becomes

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \delta(t) \quad (1\text{-}129)$$

Neglecting initial conditions and Laplace transforming, we obtain:

$$X_{part}(s) = (s^2 + 2\zeta\omega_n s + \omega_n^2) = 1 \quad (1\text{-}130)$$

or

$$X_{part}(s) = 1 / s^2 + 2\zeta\omega_n s + \omega_n^2 \quad (1\text{-}131)$$

For our example with $\zeta=1$

$$X_{part}(s) = 1/(s+8.51)^2 \quad (1\text{-}132)$$

$$x_{part}(t) = t\, e^{-8.51t} \quad (1\text{-}133)$$

and the total response is

$$x(t) = (4+66.04t)\,e^{-8.51t} + t\,e^{-8.51t} \tag{1-134}$$

with $\zeta = 1.5$

$$X_{part}(s) = 1/s^2 + 25.53s + 72.42 = 1/(s+3.26)(s+22.28) = \frac{0.0525}{s+3.26} + \frac{0.0525}{s+22.28} \tag{1-135}$$

$$x_{part}(t) = 0.0525\,e^{-3.26t} - 0.0525\,e^{-22.28t} \tag{1-136}$$

and the total response is

$$x(t) = 6.36\,e^{-3.26t} - 2.36\,e^{-22.28t} + 0.0525\,e^{-3.26t} - 0.0525\,e^{-22.28t} \tag{1-137}$$

With $\zeta = 0.5$

$$X_{part}(s) = 1/s^2 + 8.51s + 72.42 = 1/(s+4.26+j\,7.37)(s+4.26-j\,7.37)$$

$$= \frac{0.07\,e^{j90^\circ}}{s+4.26+j\,7.37} + \frac{0.07\,e^{-j90^\circ}}{s+4.26-j\,7.37} \tag{1-138}$$

$$x(t) = 0.14\,e^{-4.26t}\cos(7.37t + 1.57\text{ rad}) \tag{1-139}$$

and the total response is

$$x(t) = -7.76\,e^{-4.26t}\cos(7.37t + 2.11\text{ rad}) +$$

$$+\,0.14\,e^{-4.26t}\cos(7.37 + 1.57\text{ rad}) \tag{1-140}$$

(2) Unit Step Response

The spring mass damper is driven by the singularity input

$$F(t) = u(t) \tag{1-141}$$

The differential equation becomes:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u(t) \tag{1-142}$$

Neglecting initial conditions and Laplace transforming, we obtain:

$$X_{part}(s) \cdot (s^2 + 2\zeta\omega_n s + \omega_n^2) = s \tag{1-143}$$

or

$$X_{part}(s) = s / s^2 + 2\zeta\omega_n s + \omega_n^2 \tag{1-144}$$

For our example with $\zeta=1$

$$X_{part}(s) = \frac{s}{(s+8.51)^2} = \frac{1}{s+8.51} - \frac{8.51}{(s+8.51)^2} \tag{1-145}$$

$$x_{part}(t) = e^{-8.51t} - 8.51t\, e^{-8.51t} \tag{1-146}$$

and the total response is

$$x(t) = (4 + 66.04t)\, e^{-8.51t} + e^{-8.51t} - t^2 e^{-8.51t} \tag{1-147}$$

With $\zeta=1.5$

$$X_{part}(s) = \frac{s}{(s+3.26)(s+22.28)} = \frac{0.17}{s+3.26} \quad \frac{-1.17}{s+22.28} \tag{1-148}$$

$$x_{part}(t) = 0.17\, e^{-3.26t} - 1.17\, e^{-22.28t} \tag{1-149}$$

and the total response is

$$x(t) = 6.36\, e^{-3.26t} - 2.36\, e^{-22.28t} + 0.17\, e^{-3.26t} - 1.17\, e^{-22.28t} \tag{1-150}$$

With $\zeta=0.5$

$$X_{part}(s) = \frac{s}{(s+4.26+j7.37)(s+4.26-j7.37)} = \frac{0.58\, e^{-30.11^\circ}}{s+4.26+j7.37} + \frac{0.58\, e^{30.11^\circ}}{s+4.26-j7.37} \tag{1-151}$$

$$x_{part}(t) = 1.16\, e^{-4.26t} \cos(7.37t + 0.53 \text{ rad}) \tag{1-152}$$

and the total response is

$$x(t) = -7.76\, e^{-4.26t} \cos(7.37t + 2.11 \text{ rad}) +$$

$$+1.16\, e^{-4.26t} \cos(7.37t + 0.53 \text{ rad}) \tag{1-153}$$

## 1.6 AXES SYSTEMS

In describing the motion of our flight vehicle, we will make the following two assumptions:

First, we will establish an inertial frame of reference on the earth considering the earth to be fixed in space. In this inertial frame, Newton's laws are valid.

Second, the vehicle will be assumed to behave like a rigid body so that translational motion can be described by considering it as a particle with all its mass located at the center of mass, and rotational motion can be described by considering moments about that center of mass and constant moments of inertia.

To describe the translational and the rotational motion, one has to establish a second non-inertial axes system, fixed on the vehicle with origin at its center of mass. In this reference frame Newton's laws do not hold, in general, and we have to add additional terms to our basic fundamental equations.

It is convenient to use cartesian coordinate systems for both reference frames used.

Let's consider the translational motion first. We will assume that $Oxyz$ is the inertial reference frame and $O'x'y'z'$ the non-inertial. As in Figure 1-7 shown, $\vec{r}$ and $\vec{r}\,'$ are position vectors pointing to an object and originating from

the origins of the coordinate axes correspondingly. $\vec{r}_o$ denotes the position vector of the non-inertial origin measured from the inertial origin.

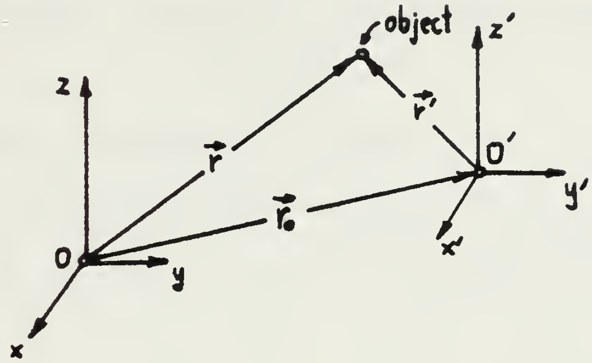


Figure 1-7

Inertial and Non-Inertial Reference Frames
Translational Motion

The relation between those three vectors is clearly

$$\vec{r} = \vec{r}\,' + \vec{r}_o$$

By differentiating twice, with respect to time, we obtain

$$\ddot{\vec{r}} = \ddot{\vec{r}}\,' + \ddot{\vec{r}}_o \qquad (1\text{-}154)$$

Now since we have assumed that Newton's laws hold on the frame $Oxyz$, we may write

$$m\ddot{\vec{r}} = \vec{F} \qquad (1\text{-}155)$$

or substituting Equation 1-154

$$m\ddot{\vec{r}}\,' = \vec{F} - m\ddot{\vec{r}}_o \qquad (1\text{-}156)$$

where $m$ is the mass of the object and $\vec{F}$ the force applied onto it.

Equation 1-156 merely says that we need the extra term $m\ddot{\vec{r}}_0$ to describe the translational motion of the object in the non-inertial frame.

If $\ddot{\vec{r}}_0 = 0$ then the frame $O'x'y'z'$ is inertial and Newton's laws hold. In this case, Equation 1-156 takes our familiar form

$$m\ddot{\vec{r}}\,' = \vec{F} \qquad (1\text{-}157)$$

which is similar to Equation 1-155.

Let's now consider the rotational case. We will assume that $\vec{r}_0 = 0$ so that $O'$ is the same as $O$, as shown in Figure 1-8.

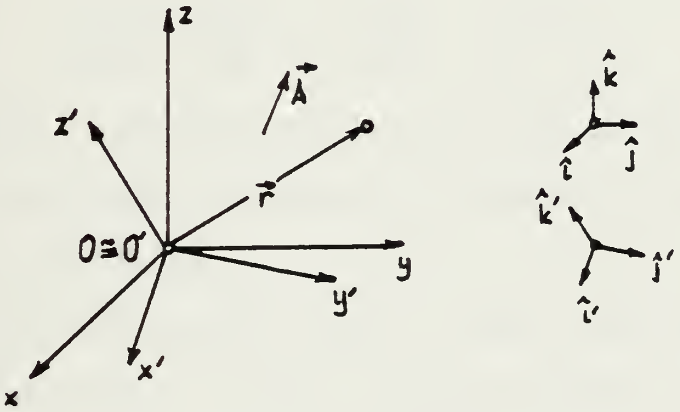


Figure 1-8

Inertial and Non-Inertial Reference Frames
Rotational Motion

The position vector $\vec{r}$ can be expressed in the inertial frame as

$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k} \qquad (1\text{-}158)$$

or in the non-inertial frame

$$\vec{r} = x'\,\hat{\imath} + y'\,\hat{\jmath} + z'\,\hat{k} \qquad (1\text{-}159)$$

where $(x, y, z)$ and $(x', y', z')$ are the corresponding axes components of $\vec{r}$.

Similarly the vector $\vec{A}$ can be expressed as

$$\vec{A} = A_x \hat{\imath} + A_y \hat{\jmath} + A_z \hat{k} \qquad (1\text{-}160)$$

or

$$\vec{A} = A_x' \hat{\imath}' + A_y' \hat{\jmath}' + A_z' \hat{k}' \qquad (1\text{-}161)$$

Differentiating Equation 1-160, 1-161 with respect to time, we obtain

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt} \hat{\imath} + \frac{dA_y}{dt} \hat{\jmath} + \frac{dA_z}{dt} \hat{k} \qquad (1\text{-}162)$$

and

$$\left(\frac{d\vec{A}}{dt}\right)' = \frac{dA_x'}{dt} \hat{\imath}' + \frac{dA_y'}{dt} \hat{\jmath}' + \frac{dA_z}{dt} \hat{k}' \qquad (1\text{-}163)$$

where $(d\vec{A}/dt)'$ denotes that we are taking the time derivative of $\vec{A}$ on the primed non-inertial system where the unit vectors $\hat{\imath}', \hat{\jmath}', \hat{k}'$ are constants.

We are interested now in the time derivative of the vector $\vec{A}$ relative to the inertial system, but we would like to differentiate Equation 1-161 instead of Equation 1-160, obtaining thus:

$$\frac{d\vec{A}}{dt} = \frac{dA_x'}{dt} \hat{\imath}' + \frac{dA_y'}{dt} \hat{\jmath}' + \frac{dA_z'}{dt} \hat{k} + A_x' \frac{d\hat{\imath}'}{dt} + A_y' \frac{d\hat{\jmath}'}{dt} + A_z' \frac{d\hat{k}'}{dt} \qquad (1\text{-}164)$$

or

$$\frac{d\vec{A}}{dt} = \left(\frac{d\vec{A}}{dt}\right)' + A_x' \frac{d\hat{\imath}'}{dt} + A_y' \frac{d\hat{\jmath}'}{dt} + A_z' \frac{d\hat{k}'}{dt} \qquad (1\text{-}165)$$

It can be shown that a more convenient expression for the three last terms is given by

$$A_x' \frac{d\hat{i}'}{dt} + A_y' \frac{d\hat{j}'}{dt} + A_z' \frac{d\hat{k}'}{dt} = \vec{\omega} \times \vec{A} \qquad (1\text{-}166)$$

where $\vec{\omega}$ is the angular velocity vector of the primed system which is assumed to rotate about an axis having the direction of $\vec{\omega}$, through the origin, and with angular velocity $\vec{\omega}$, relative to the unprimed inertial frame.

For proof, assume for a moment that $\vec{A}$ is a vector "at rest" on the primed system so that

$$(d\vec{A}/dt)' = 0 \qquad (1\text{-}167)$$

We shall show that

$$d\vec{A}/dt = \vec{\omega} \times \vec{A} \qquad (1\text{-}168)$$

Using the time derivative definition, we can write:

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t} = \vec{\omega} \times \vec{A} \qquad (1\text{-}169)$$

Figure 1-9 shows two successive positions of the vector $\vec{A}$ which rotates with angular velocity $\vec{\omega}$ with respect to the inertial frame, so that the tip of the vector traces a circle perpendicular to the direction of $\vec{\omega}$.

To show Equation 1-169, we must show that the directions and magnitudes of the vectors on the left and right-hand side are the same.

Notice that the direction of the vector $d\vec{A}/dt$, by Equation 1-169, is the same as the direction of the vector $\vec{A}(t+\Delta t) - \vec{A}(t)$, which is the same as the direction of the vector $\vec{\omega} \times \vec{A}$ by looking at Figure 1-9.

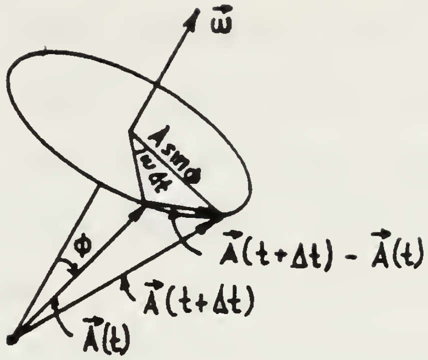


Figure 1-9

Time Derivative of a Rotating Vector

The magnitude of the left-hand side of Equation 1-169 is:

$$\lim_{\Delta t \to 0} \frac{|\vec{A}(t+\Delta t) - \vec{A}(t)|}{\Delta t} = \lim_{\Delta t \to 0} \frac{(A\sin\phi)(\omega\Delta t)}{\Delta t} \qquad (1\text{-}170)$$

or

$$\left|\frac{d\vec{A}}{dt}\right| = A\omega\sin\phi \qquad (1\text{-}171)$$

which is the same as the magnitude of the right-hand side vector, $|\vec{\omega} \times \vec{A}|$.

Hence, both directions and magnitudes are the same, so Equation 1-168 holds and if $\vec{A}$ is not stationary with respect to the non-inertial system, we have according to Equation 1-165

$$\frac{d\vec{A}}{dt} = \left(\frac{d\vec{A}}{dt}\right)' + \vec{\omega} \times \vec{A} \tag{1-172}$$

which says that the time derivative of a vector $\vec{A}$ relative to an inertial frame equals the time derivative of the same vector measured at the non-inertial frame plus the term $\vec{\omega}\times\vec{A}$ where $\vec{\omega}$ is the angular velocity of the non-inertial as seen by the inertial frame.

Combining translational and rotational motions, one can easily arrive at the equation:

$$\frac{d\vec{A}}{dt} = \left(\frac{d\vec{A}}{dt}\right)' + \vec{\omega} \times \vec{A} + \dot{\vec{r}}_0 \tag{1-173}$$

where $\dot{\vec{r}}_0$ is the linear velocity of the non-inertial with respect to the inertial frame.

One step further, differentiating once again Equation 1-172, with respect to time, we obtain

$$\frac{d^2\vec{A}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{A}}{dt}\right)' + \vec{\omega} \times \frac{d\vec{A}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{A} =$$

$$= \left(\frac{d^2\vec{A}}{dt^2}\right)' + \vec{\omega} \times \left(\frac{d\vec{A}}{dt}\right)' + \vec{\omega} \times \left\{\left(\frac{d\vec{A}}{dt}\right)' + \vec{\omega} \times \vec{A}\right\} + \frac{d\vec{\omega}}{dt} \times \vec{A} \tag{1-174}$$

or

$$\frac{d^2\vec{A}}{dt^2} = \left(\frac{d^2\vec{A}}{dt^2}\right)' + 2\,\vec{\omega} \times \left(\frac{d\vec{A}}{dt}\right)' + \vec{\omega} \times \left(\vec{\omega} \times \vec{A}\right) + \frac{d\vec{\omega}}{dt} \times \vec{A} \tag{1-175}$$

which says that the second time derivative of a vector $\vec{A}$ relative to an inertial frame equals the second time derivative of the same vector measured at the non-inertial frame, plus some extra terms involving $\vec{\omega}$, $\vec{A}$ , $d\vec{\omega}/dt$ , and

$(d\vec{A}/dt)'$ where $\vec{\omega}$ is again the angular velocity of the non-inertial as seen by the inertial frame.

Combining translational and rotational motions one can easily arrive at the equation

$$\frac{d^2\vec{A}}{dt^2} = \left(\frac{d^2\vec{A}}{dt^2}\right)' + 2\vec{\omega}\times\left(\frac{d\vec{A}}{dt}\right)' + \vec{\omega}\times(\vec{\omega}\times\vec{A}) + \frac{d\vec{\omega}}{dt}\times\vec{A} + \ddot{\vec{r}}_0 \qquad (1\text{-}176)$$

where $\ddot{\vec{r}}_0$ is the linear acceleration of the non-inertial with respect to the inertial frame.

A special significance of Equations 1-173 and 1-176 is given by substituting $\vec{A}$ with the position vector $\vec{r}$ of an object. They are taking now the form

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\omega}\times\vec{r} + \dot{\vec{r}}_0 \qquad (1\text{-}177)$$

or

$$\vec{V} = (\vec{V})' + \vec{\omega}\times\vec{r} + \dot{\vec{r}}_0 \qquad (1\text{-}178)$$

where $\vec{V}$ is the velocity of the object, and

$$\frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2\vec{r}}{dt^2}\right)' + 2\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)' + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \frac{d\vec{\omega}}{dt}\times\vec{r} + \ddot{\vec{r}}_0 \qquad (1\text{-}179)$$

or

$$\vec{\alpha} = (\vec{\alpha})' + 2\vec{\omega}\times(\vec{V})' + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r} + \vec{\alpha}_0 \qquad (1\text{-}180)$$

where $\vec{\alpha}$ is the acceleration of the object.

One can recognize that

$$2\vec{\omega}\times(\vec{V})' = \text{corriolis acceleration} \qquad (1\text{-}181)$$

and exists if the object has a velocity $(\vec{V})'$ with respect to the non-inertial frame. Also,

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \text{centripedal acceleration} \qquad (1\text{-}182)$$

which is always directed toward the axis of rotation and is perpendicular to that axis, as shown in Figure 1-10.

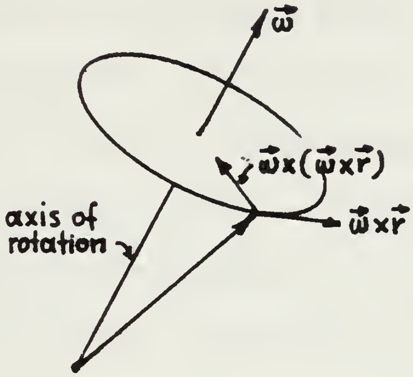

$\vec{\omega}$
$\vec{\omega}\times(\vec{\omega}\times\vec{r})$
axis of rotation
$\vec{\omega}\times\vec{r}$

Figure 1-10

Centripetal Acceleration

Since we have assumed that the unprimed system is inertial, we may write

$$m\vec{a} = \vec{F} \tag{1-183}$$

and substituting Equation 1-180

$$m(\vec{a})' = \vec{F} - 2m[\vec{\omega}\times(\vec{V})'] - m[\vec{\omega}\times(\vec{\omega}\times\vec{r})] - m(\dot{\vec{\omega}}\times\vec{r}) - m\vec{a}_o \tag{1-184}$$

where $m$ is the mass of the object and $\vec{F}$ the force applied to it.

Equation 1-184 merely says that we need four extra terms to describe the translational and the rotational motion in a non-inertial frame.

It is clear that the forces:

$$2m[\vec{\omega}\times(\vec{V})'] = \text{corriolis force} \qquad (1\text{-}185)$$

$$m[\vec{\omega}\times(\vec{\omega}\times\vec{r})] = \text{centripedal force} \qquad (1\text{-}186)$$

$$m(\dot{\vec{\omega}}\times\vec{r}) = \text{Euler force} \qquad (1\text{-}187)$$

$$m\vec{a}_o = \text{D' Alembert force} \qquad (1\text{-}188)$$

are not true forces. They only exist due to the fact that our object is moving on a non-inertial frame.

According to our first assumption that the earth is the inertial frame of reference, we consider that it is rotating with fixed angular velocity $\vec{\omega}$ so that we can write Equation 1-184 as:

$$m(\vec{a})' = \vec{F} - 2m\,\vec{\omega}\times(\vec{V})' - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}) - m\vec{a}_o \qquad (1\text{-}189)$$

We have seen, up to now, the effects of considering a non-inertial frame of reference and our next step will be to orient our flight vehicle in that frame. The reason for selecting a non-inertial frame will be greatly appreciated when one will see how easy it is to formulate equations of motion of a body attached in this frame and moving with it.

For flight control applications, it is appropriate to select the origin of the axis system on the center of mass of the vehicle.

From then on we will orient the nose of the vehicle towards the positive x-axis, which will also be a symmetry axis. The positive y-axis will point to the right and the positive z-axis will point downwards having the direction of gravity, if one considers straight level flight as

Figure 1-11 shows, so that the xz plane will be a plane of symmetry.

Vector quantities associated with this axis system are shown in Figure 1-11 and are described as follows:

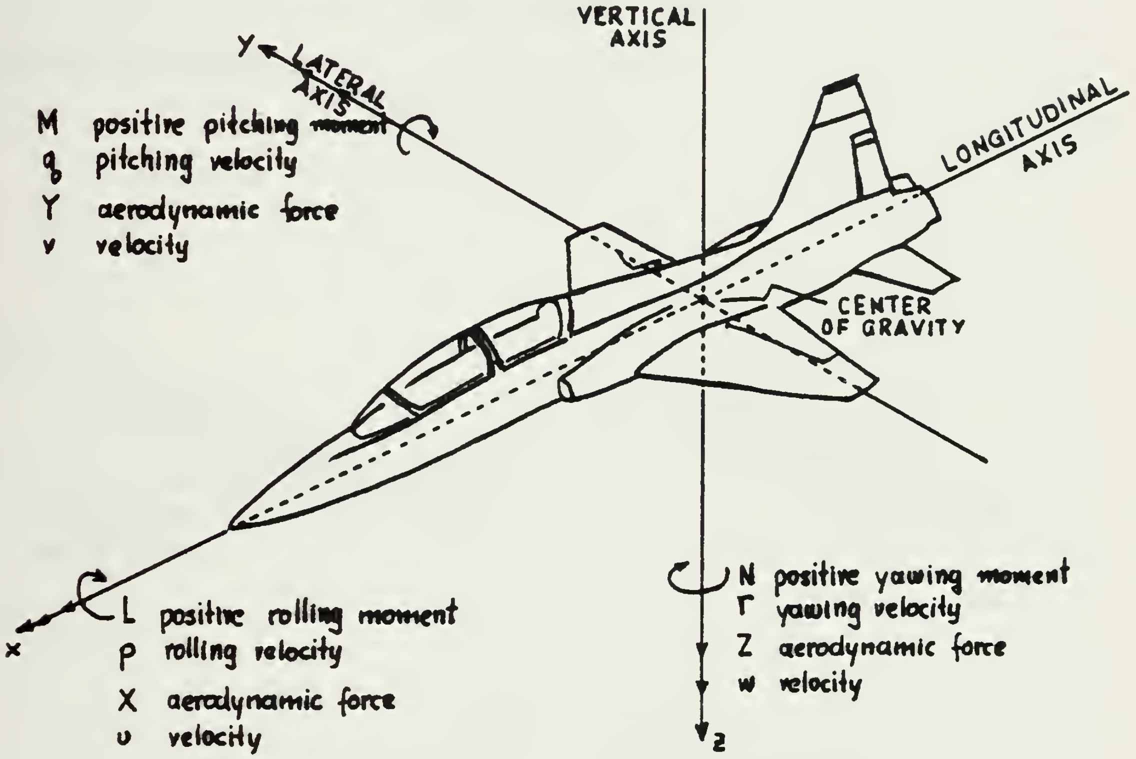


Figure 1-11

Axes System

Velocity of the center of mass with respect to the inertial frame measured on this axis system

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k} \tag{1-190}$$

Angular velocity about the center of mass

$$\vec{\omega} = p\hat{i} + q\hat{j} + r\hat{k} \tag{1-191}$$

Aerodynamic force acting on the center of mass

$$\vec{F} = X\hat{\imath} + Y\hat{\jmath} + Z\hat{k} \qquad (1\text{-}192)$$

Moment about the center of mass

$$\vec{G} = L\hat{\imath} + M\hat{\jmath} + N\hat{k} \qquad (1\text{-}193)$$

Directions as shown in Figure 1-11

Also, we can define: (see Figure 1-12)

Angle of attack:

$$\alpha = \tan^{-1}(w/u) \qquad (1\text{-}194)$$

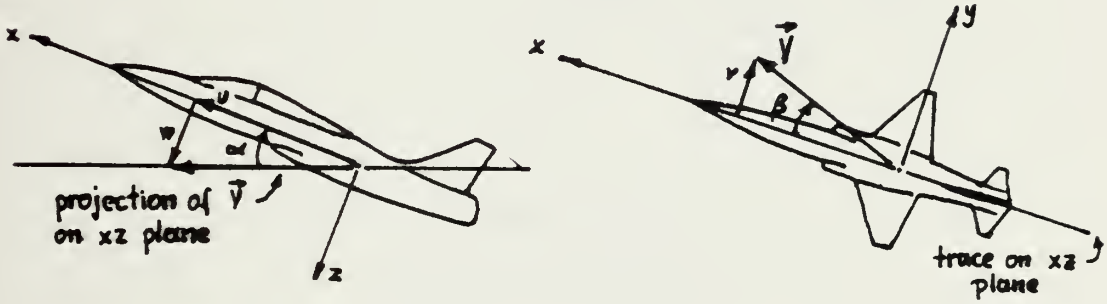


Figure 1-12

Angle of Attack and Angle of Sideslip

Angle of sideslip

$$\beta = \sin^{-1}(v/V) \qquad (1\text{-}195)$$

The reference axis system has been adopted by B. Etkin, D. McRuer and others, and is usually referred to as body axis system (see references). The notation used is quite common on most references.

Another axes system adopted by C. Perkins is shown in Figure 1-13. It is usually referred to as wind or stability axes system.

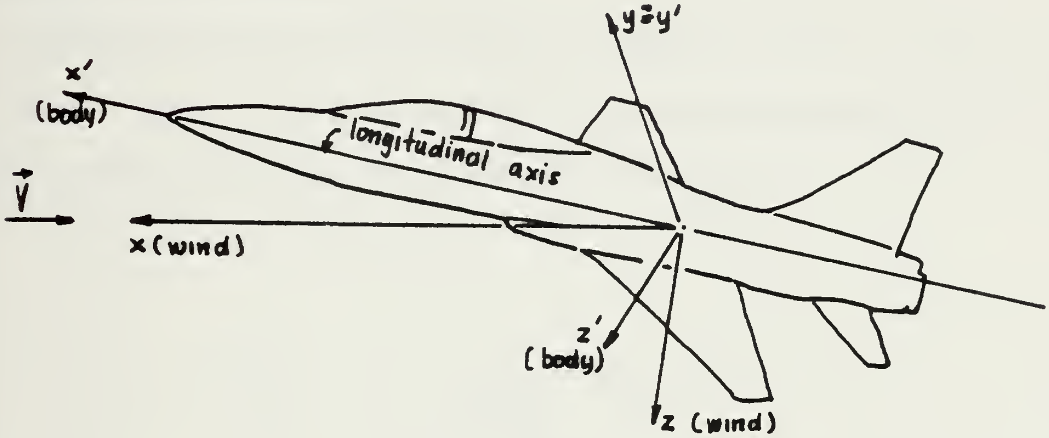


Figure 1-13

Wind Axes System

In this system the x-axis points to the opposite direction to that of the relative wind. Notice that in the undisturbed flight the body axes coincided with the wind axes, but in the disturbed flight that may or may not happen.

## 1.7 RIGID BODY DYNAMICS

Rigid body is a system of particles in which the distance between any two of them is not changing regardless of the acting forces. For our flight vehicle the rigid body

assumption implies that as far as its motion is concerned in three dimentions, only negligible errors are introduced.

We will consider first the planar motion of rigid body where we will find the kinetic energy and the angular momentum expressions.

Assume that the rigid body in Figure 1-14 can move in the xy-plane and can rotate about an axis perpendicular to that plane passing through the center of mass c.

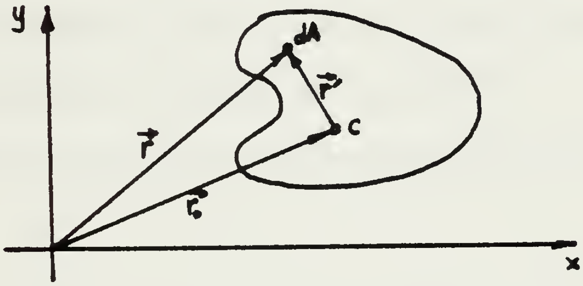


Figure 1-14

Rigid Body - Planar Motion

We denote $\vec{r}_0$ , the position vector of the center of mass c, $\vec{r}$ the position vector of any particle on the rigid body and $\vec{r}'$ the vector from the center of mass to that particle.

Usually, the mass of the rigid body is not uniformally distributed over the entire volume V. It is convenient to

think for a moment that the entire volume of the rigid body containing a mass $m$, is projected on the xy-plane on an area A. Let's denote $\eta$ the mass per unit projected area of the body. It is clearly seen that the summation over the entire area of all those masses $\eta dA$ gives the total mass of the body. This mathematically is expressed as:

$$m = \int_A \eta dA \tag{1-196}$$

From Figure 1-14 we have:

$$\vec{r} = \vec{r}_0 + \vec{r}\,' \tag{1-197}$$

For a differential mass $\eta dA$ at which the $\vec{r}$ vector points, the kinetic energy is expressed as $\frac{1}{2}\eta dA\,\dot{r}^2$ . Integration over the entire area A gives the total kinetic energy of the rigid body.

$$m = \frac{1}{2}\int_A \dot{r}^2 \eta dA \tag{1-198}$$

The quantity $\dot{r}^2$ according to Equation 1-197 is expressed as

$$\dot{r}^2 = \dot{\vec{r}}\cdot\dot{\vec{r}} = (\dot{\vec{r}}_0 + \dot{\vec{r}}\,')\cdot(\dot{\vec{r}}_0 + \dot{\vec{r}}\,') = \dot{r}_0^{\,2} + \dot{r}'^{\,2} + 2\dot{\vec{r}}_0\dot{\vec{r}}\,' \tag{1-199}$$

Then Equation 1-198 becomes

$$K = \frac{1}{2}\dot{r}_0^{\,2}\int_A \eta dA + \frac{1}{2}\int_A \dot{r}'^{\,2}\eta dA + \dot{\vec{r}}_0\int_A \dot{\vec{r}}\,'\eta dA \tag{1-200}$$

The last term is zero since

$$\int_A \dot{\vec{r}}\,'\eta dA = \int_A \dot{\vec{r}}\eta dA - \dot{\vec{r}}_0\int_A \eta dA = \dot{\vec{r}}_0 m - \dot{\vec{r}}_0 m = 0 \tag{1-201}$$

We now express the velocity $\dot{\vec{r}}'$ as a sum of two velocities:

$$\dot{\vec{r}}' = \frac{dr'}{dt}\hat{r} + r\dot{\theta}\hat{\vartheta} \qquad (1\text{-}202)$$

where $dr'/dt$ is the radial velocity, $\dot{\theta}$ is the angular velocity of the body, $r'\dot{\theta}$ the tangential velocity of the particle , $\hat{r}$ and $\hat{\vartheta}$ denote the corresponding unit vectors in polar coordinates.

Noticing that $dr'/dt = 0$, Equation 1-200 is written

$$K = \frac{1}{2}m\dot{r}_o^2 + \frac{1}{2}\dot{\theta}^2\int_A r'^2 \eta \, dA \qquad (1\text{-}203)$$

We define

$$I = \int_A r'^2 \eta \, dA \qquad (1\text{-}204)$$

the moment of inertia of the body about an axis perpendicular to the plane of motion through the center of mass. Then

$$K = \frac{1}{2}m\dot{r}_o^2 + \frac{1}{2}I\dot{\theta}^2 \qquad (1\text{-}205)$$

This equation merely says that, in addition to the term $\frac{1}{2}m\dot{r}_o^2$ that describes the translational motion, we need the extra term $\frac{1}{2}I\dot{\theta}^2$ which describes the rotational motion of the body around an axis perpendicular to the plane of motion through the center of mass.

The angular momentum of the same differential mass $\eta dA$ about the origin is expressed as

$$\vec{r} \times \eta \, dA \, \dot{\vec{r}} \qquad (1\text{-}206)$$

Integration over the entire A gives the total angular momentum of the rigid body.

$$\vec{h} = \int_A (\vec{r} \times \dot{\vec{r}})\, \eta\, dA \tag{1-207}$$

The cross product term can be written

$$\vec{r} \times \dot{\vec{r}} = (\vec{r}_0 + \vec{r}\,') \times (\vec{r}_0 + \vec{r}\,') = \vec{r}_0 \times \dot{\vec{r}}_0 + \vec{r}\,' \times \dot{\vec{r}}\,' + \dot{\vec{r}}\,' \times \dot{\vec{r}}_0 + \vec{r}_0 \times \vec{r}\,' \tag{1-208}$$

Then Equation 1-207 becomes

$$\vec{h} = \vec{r}_0 \times \dot{\vec{r}}_0 \int_A \eta\, dA + \int_A (\vec{r}\,' \times \dot{\vec{r}}\,')\, \eta\, dA + \dot{\vec{r}}_0 \times \int_A \vec{r}\,' \eta\, dA + \vec{r}_0 \times \int_A \dot{\vec{r}}\,' \eta\, dA \tag{1-209}$$

In this expression the integral of the last term is zero as Equation 1-201 shows. By a similar argument the third term is also zero. Noticing that $\dot{\vec{r}}\,' = r' \dot{\theta}\, \hat{\vartheta}$ , Equation 1-209 becomes

$$\vec{h} = (\vec{r}_0 \times \dot{\vec{r}}_0)\, m + \dot{\theta} \int_A r' (\vec{r}\,' \times \hat{\vartheta})\, \eta\, dA \tag{1-210}$$

If we use the center of mass at the origin ($\vec{r}_0 = 0$) we are left with:

$$\vec{h} = \dot{\theta} \int_A r (\vec{r} \times \hat{\vartheta})\, \eta\, dA \tag{1-211}$$

The cross product in the integral indicates that the angular momentum vector is perpendicular to the plane of motion. Since $|\vec{r} \times \hat{\vartheta}| = r$, the magnitude of the angular momentum is given by:

$$h = \dot{\theta} \int_A r^2 \eta\, dA = I \dot{\theta} \tag{1-212}$$

So the magnitude of the angular momentum of a rigid body rotating in a plane, about an axis, through the center of mass, is given by Equation 1-212. The angular momentum vector points in the direction of the axis of rotation that can be easily found by the right hand rule.

In summary, we have seen that the motion of a rigid body in planar motion can be described by a translational motion of the center of mass, plus a rotational motion about an axis through the center of mass.

We will now look at the motion of a rigid body in a three dimensional space. It generally has six degrees of freedom if not constrained. Hence, we need six coordinates to specify the motion. The three of them can describe the translational motion of the center of mass of the body and the remaining three, the rotational motion.

We consider two coordinate systems. An unprimed system, as shown in Figure 1-15, whose origin is O, for simplicity is fixed on the rigid body but its axes do not rotate with respect to an inertial frame, and a primed system whose origin and axes are fixed on the rigid body.

We also assume that the rigid body has a mass distribution per unit volume (density) which, in general, is a function of the position vector $\vec{r}$ to a particle on the rigid body having a differential volume $dv$. The total mass is given by:

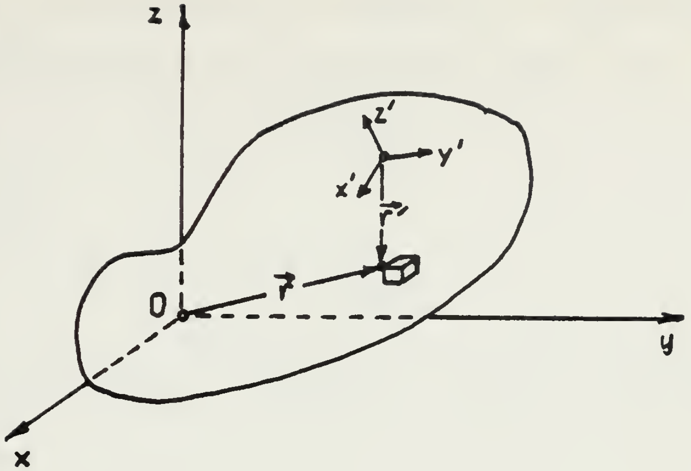


Figure 1-15

Rigid Body - Space Rotational Motion

$$m = \int_v \rho(\vec{r})\,dv \tag{1-213}$$

where $\rho(\vec{r})$ is the density as a function of $\vec{r}$ and $dv$ a small differential volume where $\vec{r}$ points to.

For the differential volume $dv$ the angular momentum for rotation about the origin of the unprimed system is given by

$$\vec{r} \times \left[\, \rho(\vec{r})\,dv \cdot \dot{\vec{r}}\,\right] \tag{1-214}$$

Integration over the entire volume gives the total angular momentum:

$$\vec{h} = \int_v (\vec{r} \times \dot{\vec{r}})\, \rho(\vec{r})\,dv \tag{1-215}$$

where $\vec{r}, \dot{\vec{r}}$ are measured on the unprimed system.

Let $\vec{\omega}$ the angular velocity of the rigid body about the origin of the unprimed system and $\vec{r}'$ the position vector from the primed system.

It is more convenient to express $\dot{\vec{r}}$, which can be seen as the velocity measured on an inertial frame, as

$$\dot{\vec{r}} = \dot{\vec{r}}' + \vec{\omega} \times \vec{r} \tag{1-216}$$

Notice that

$$\dot{\vec{r}}' = \frac{dr'}{dt}\,\hat{r}' + r'\dot{\theta}\,\hat{\vartheta}' \tag{1-217}$$

where $\hat{r}'$ and $\hat{\vartheta}'$ are unit vectors in polar coordinates of the primed system.

It is clear that $dr'/dt = 0$ according to rigid body assumption, and $\dot{\theta} = 0$ because there is no relative rotation of the differential volume $dv$ that can be seen by the primed system. In other words, $\vec{r}'$ is a vector at rest in the primed system so that we can write

$$\dot{\vec{r}} = \vec{\omega} \times \vec{r} \tag{1-218}$$

Substituting in Equation 1-215), we obtain

$$\vec{h} = \int_v \vec{r} \times (\vec{\omega} \times \vec{r})\, \rho(\vec{r})\, dv \tag{1-219}$$

Using now the tripple product identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \tag{1-220}$$

we can write

$$\vec{r} \times (\vec{\omega} \times \vec{r}) = r^2 \vec{\omega} - (\vec{\omega} \cdot \vec{r})\vec{r} \tag{1-221}$$

so that Equation 1-215 becomes

$$\vec{h} = \int_v \left[ r^2 \vec{\omega} - (\vec{\omega} \cdot \vec{r})\vec{r} \right] \rho(\vec{r})\, dv \tag{1-222}$$

In component form we can write

$$h_i = \int_v \left[ r^2 \omega_i - (\vec{\omega} \cdot \vec{r})\, r_i \right] \rho(\vec{r})\, dv \tag{1-223}$$

where $i = 1$, 2, or 3 denotes the x, y, or z component respectively. The dot product is expressed as:

$$\vec{\omega}\cdot\vec{r} = \omega_1 r_1 + \omega_2 r_2 + \omega_3 r_3 = \sum_{j=1}^{3} r_j \omega_j \qquad (1\text{-}224)$$

That permits us to write

$$h_i = \sum_{j=1}^{3} \int_v \left[ r^2 \delta_{ij} - r_i r_j \right] \omega_j \, \rho(\vec{r})\, dv \qquad (1\text{-}225)$$

where $\delta_{ij} = 1$ if $i=j$ and $\delta_{ij} = 0$ if $i \neq j$ because

$$\sum_{j=1}^{3} \delta_{ij}\,\omega_j = \delta_{i1}\,\omega_1 + \delta_{i2}\,\omega_2 + \delta_{i3}\,\omega_3 = \omega_i \qquad (1\text{-}226)$$

We can now define

$$I_{ij} = \int_v \left[ r^2 \delta_{ij} - r_i r_j \right] \rho(\vec{r})\, dv \qquad (1\text{-}227)$$

as the $ij$ th component of the inertia tensor I about the origin of the unprimed system.

So finally, we express the $i$ th component of the angular momentum as:

$$h_i = \sum_{j=1}^{3} I_{ij}\,\omega_j \qquad (1\text{-}228)$$

In the expanded matrix form this gives:

$$\begin{vmatrix} h_x \\ h_y \\ h_z \end{vmatrix} = \begin{vmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{vmatrix} \cdot \begin{vmatrix} \omega_x \\ \omega_y \\ \omega_z \end{vmatrix} \qquad (1\text{-}229)$$

This result is compared with the result of the planar motion case that is expressed by Equation 1-212, where the inertia scalar term is expressed now as a matrix.

The components of the inertia tensor are given by Equation 1-227 in cartesian coordinates as:

$$I_{xx} = \iiint \left[ (x^2+y^2+z^2) - x^2 \right] \rho(\vec{r})\, dx\,dy\,dz =$$

$$= \iiint (y^2+z^2)\, \rho(\vec{r})\, dx\,dy\,dz \qquad (1\text{-}230)$$

Similarly,

$$I_{yy} = \iiint (x^2+z^2)\, \rho(\vec{r})\, dx\,dy\,dz \qquad (1\text{-}231)$$

$$I_{zz} = \iiint (x^2+y^2)\, \rho(\vec{r})\, dx\,dy\,dz \qquad (1\text{-}232)$$

Also,

$$I_{xy} = I_{yx} = -\iiint xy\, \rho(\vec{r})\, dx\,dy\,dz \qquad (1\text{-}233)$$

$$I_{xz} = I_{zx} = -\iiint xz\, \rho(\vec{r})\, dx\,dy\,dz \qquad (1\text{-}234)$$

Example: Calculate the inertia tensor of a cube of side a, mass m and uniform density $\rho$ about one of its corners (Figure 1-16).

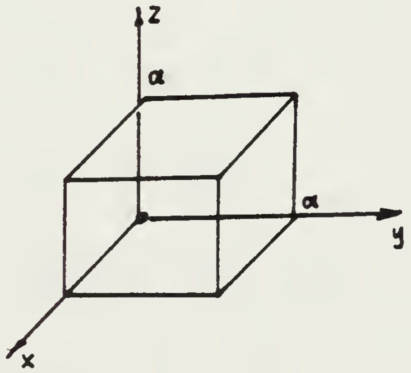


Figure 1-16

Inertia Tensor-Calculation of a Cube

$$I_{xx} = \rho \iiint (y^2+z^2)\, dx\,dy\,dz = \frac{m}{a^3} \int_0^a \int_0^a \int_0^a (y^2+z^2)\, dz\,dy\,dx =$$

$$= \frac{m}{a^3} \int_0^a \int_0^a \left( y^2 a + \frac{a^3}{3} \right) dy\,dx = \frac{m}{a^3} \int_0^a \left( \frac{a^3}{3} a + \frac{a^3}{3} a \right) dx =$$

$$= \frac{m}{a^3} \left( \frac{a^4}{3} a + \frac{a^4}{3} a \right) = \frac{m}{a^3} \cdot \frac{2a^5}{3} = \frac{2}{3} m a^2 \qquad (1\text{-}235)$$

Similarly,

$$I_{yy} = I_{zz} = \frac{2}{3} m a^2 \qquad (1\text{-}236)$$

$$I_{xy} = -\rho \iiint xy\, dx\,dy\,dz = -\frac{m}{a^3} \int_0^a \int_0^a \int_0^a xy\, dz\,dy\,dx =$$

$$= \frac{m}{a^3} \int_0^a \int_0^a xya\, dy\,dx = -\frac{m}{a^3} \int_0^a x \frac{a^2}{2} a\, dx =$$

$$= -\frac{m}{a^3} \cdot \frac{a^3}{2} \cdot \frac{a^2}{2} = -\frac{1}{4} m a^2 \qquad (1\text{-}237)$$

Similarly,

$$I_{xz} = -\frac{1}{4} m a^2 \qquad (1\text{-}238)$$

so that the inertia tensor is written:

$$\underline{I} = m a^2 \begin{vmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{vmatrix} \qquad (1\text{-}239)$$

The eigenvalues and the eigenvectors of an inertia tensor have a very special significance. The eigenvectors determine the directions of the so-called principal axes of a rigid body that pass through the point where the inertia tensor is calculated (usually the center of mass). The corresponding eigenvalues are called the principal moments of inertia.

It can be shown that if a rigid body rotates about one of its principal axes, its angular momentum about the point which the inertia tensor is calculated, is parallel to its angular velocity.

For proof, let $\vec{\beta}$ a principal axis, and B the corresponding principal moment of inertia. The eigenvalue problem equation which has to be satisfied is:

$$I\vec{\beta} = B\vec{\beta} \tag{1-240}$$

or

$$I_{ij}\beta_j = B\beta_i \tag{1-241}$$

Having rotation about a principal axis simply means that $\vec{\omega}$ has the direction of the principal axis which is expressed as:

$$\vec{\omega} = c\vec{\beta} \tag{1-242}$$

where C is a constant. In fact, if $\vec{\beta}$ is unit vector, C is the magnitude of the angular velocity $\vec{\omega}$.

Now the ith component of the angular momentum becomes

$$h_i = I_{ij}\omega_j = I_{ij}(c\beta_j) = c\,I_{ij}\beta_j = cB\beta_i = B(c\beta_i) \tag{1-243}$$

and finally,

$$h_i = B\omega_i \tag{1-244}$$

In vector form

$$\vec{h} = \mathcal{B}\,\vec{\omega} \tag{1-245}$$

which proves that $\vec{h}$ is parallel to $\vec{\omega}$.

It can also be shown that

1. Any plane of symmetry of a rigid body is perpendicular to a principal axis.

2. Any axis of symmetry of a rigid body is a principal axis.

The density $\rho$ may not be constant. For example, if the xy plane is a plane of symmetry, that simply means that

$$\rho(-x,y,z) = \rho(x,y,z) \tag{1-246}$$

and if the z-axis is an axis of symmetry, that simply means that

$$\rho(-x,-y,z) = \rho(x,y,z) \tag{1-247}$$

3. Since the inertia tensor of a rigid body is symmetric, (i.e. $I_{xy} = I_{yx}$ etc.), the principal axes are always orthogonal to each other (Important result from Linear Algebra).

4. If two or three eigenvalues of a symmetric tensor are the same, there exist two or three orthogonal eigenvectors associated with that eigenvalue.

Example: The inertia tensor of a rigid body has been calculated with the reference point at the center of mass and has been found to be

$$I = \begin{vmatrix} 15 & 0 & -10 \\ 0 & 25 & 0 \\ -10 & 0 & 30 \end{vmatrix}$$

Find the principal axes and the corresponding principal moments of inertia.

Solution:

Let $\vec{\lambda}$ denote the eigenvector (principal axis) and $\Lambda$ the corresponding eigenvalue (principal moment of inertia).

The following eigenvalue equation has to be satisfied

$$I\vec{\lambda} = \Lambda\vec{\lambda} \tag{1-248}$$

from which

$$(\Lambda - I)\vec{\lambda} = 0 \Rightarrow \begin{vmatrix} \Lambda-15 & 0 & 10 \\ 0 & \Lambda-25 & 0 \\ 10 & 0 & \Lambda-30 \end{vmatrix} = 0 \tag{1-249}$$

or

$$(\Lambda-15)(\Lambda-25)(\Lambda-30) + 10\left[-10(\Lambda-25)\right] = 0 \tag{1-250}$$

$$\Rightarrow (\Lambda-25)\left[(\Lambda-15)(\Lambda-30) - 100\right] = 0 \tag{1-251}$$

$$\Rightarrow (\Lambda-25)(\Lambda-35)(\Lambda-10) = 0 \tag{1-252}$$

Hence, the principal moments of inertia are:

$$\Lambda_1 = 10 \qquad \Lambda_2 = 25 \qquad \Lambda_3 = 35 \tag{1-253}$$

To find the eigenvectors:

For $\Lambda_1 = 10$ Equation 1-249 becomes

$$\begin{vmatrix} -5 & 0 & 10 \\ 0 & -15 & 0 \\ 10 & 0 & -20 \end{vmatrix} \vec{\lambda}_1 = 0 \tag{1-254}$$

which yields

$$\begin{aligned} -5\lambda_{1x} + 10\lambda_{1z} &= 0 &\Rightarrow\quad& \lambda_{1x} = 2\lambda_{1z} \\ -15\lambda_{1y} &= 0 &\Rightarrow\quad& \lambda_{1y} = 0 \\ 10\lambda_{1x} - 20\lambda_{1z} &= 0 &\Rightarrow\quad& \lambda_{1x} = 2\lambda_{1z} \end{aligned} \tag{1-255}$$

The eigenvector that satisfied Equation 1-254 is

$$\vec{\lambda}_1 = \left|\begin{matrix} 2 \\ 0 \\ 1 \end{matrix}\right| \tag{1-256}$$

It is more convenient to express the eigenvector as a unit vector

$$\vec{\lambda}_1 = \left|\begin{matrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{matrix}\right| \tag{1-257}$$

Similarly, we find that

$$\vec{\lambda}_2 = \left|\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right| \quad \text{assosiated with} \quad \Lambda_2 = 25 \tag{1-258}$$

and

$$\vec{\lambda}_3 = \left|\begin{matrix} 1/\sqrt{5} \\ 0 \\ -2/\sqrt{5} \end{matrix}\right| \quad \text{assosiated with} \quad \Lambda_3 = 35 \tag{1-259}$$

It is important to note that the above eigenvectors are orthogonal since they are coming from a symmetric tensor.

Since now the principal axes of a rigid body are orthogonal to one another, or can be chosen orthogonal, it is

very convenient to have them coincide with the primed system axes, the origin of which can be moved to the origin O of the the unprimed system. The importance of this arrangement will be shown a little later.

The kinetic energy of the differential volume dv is $\frac{1}{2}(\dot{\vec{r}}\cdot\dot{\vec{r}})\rho(\vec{r})dv$. Integration over the entire volume gives the total kinetic energy:

$$K = \frac{1}{2}\int_v (\dot{\vec{r}}\cdot\dot{\vec{r}})\,\rho(\vec{r})\,dv \tag{1-260}$$

Substituting

$$\dot{\vec{r}} = \vec{\omega}\times\vec{r} \tag{1-261}$$

we obtain

$$K = \frac{1}{2}\int_v (\vec{\omega}\times\vec{r})(\vec{\omega}\times\vec{r})\,\rho(\vec{r})\,dv \tag{1-262}$$

Using Equation 1-224, 1-226 and the vector identity

$$(\vec{A}\times\vec{B})(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}) \tag{1-263}$$

we can write

$$\begin{aligned}(\vec{\omega}\times\vec{r})(\vec{\omega}\times\vec{r}) &= r^2\left(\sum_{i=1}^{3}\delta_{ij}\,\omega_i\right)\left(\sum_{j=1}^{3}\delta_{ij}\,\omega_j\right) - \left(\sum_{i=1}^{3} r_i\,\omega_i\right)\left(\sum_{j=1}^{3} r_j\,\omega_j\right)\\ &= \sum_{i=1}^{3}\left\{\sum_{j=1}^{3}\left[r^2\delta_{ij} - r_i r_j\right]\omega_j\right\}\omega_i\end{aligned} \tag{1-264}$$

so that

$$K = \frac{1}{2}\sum_{i=1}^{3}\left\{\sum_{j=1}^{3}\int_v \left[r^2\delta_{ij} - r_i r_j\right]\omega_j\right\}\omega_i \tag{1-265}$$

and finally,

$$K = \frac{1}{2} \sum_{i=1}^{3} \left( \sum_{j=1}^{3} I_{ij} \omega_j \right) \omega_i = \frac{1}{2} \sum_{i=1}^{3} h_i \omega_i = \frac{1}{2} (\vec{h} \cdot \vec{\omega}) \quad (1\text{-}266)$$

Notice that if the body is rotating about a principal axis ( $\vec{h} \parallel \vec{\omega}$ ) its kinetic energy is maximum $K = \frac{1}{2} h\omega$. That happens when the rigid body rotates about a principal axis.

This compares with the result of the planar motion case that is expressed by Equation 1-205, if we neglect the translational motion.

We are now in a position to formulate the rigid body equations of motion.

The translational motion equation is given by the equation of motion of the center of mass of the rigid body

$$m\ddot{\vec{r}}_0 = \vec{F} \quad (1\text{-}267)$$

or

$$m\left( \frac{d\vec{V}}{dt} + \vec{\omega} \times \vec{V} \right) = \vec{F} \quad (1\text{-}268)$$

The rotational motion about the point (O) which is conveniently chosen as the center of mass (c), as Figure 1-17 shows, is given by

$$\frac{d\vec{h}}{dt} = \vec{G} \quad (1\text{-}269)$$

where $\vec{G}$ is the net external moment about the center of mass (c) and $\vec{h}$ is given by Equation 1-228.

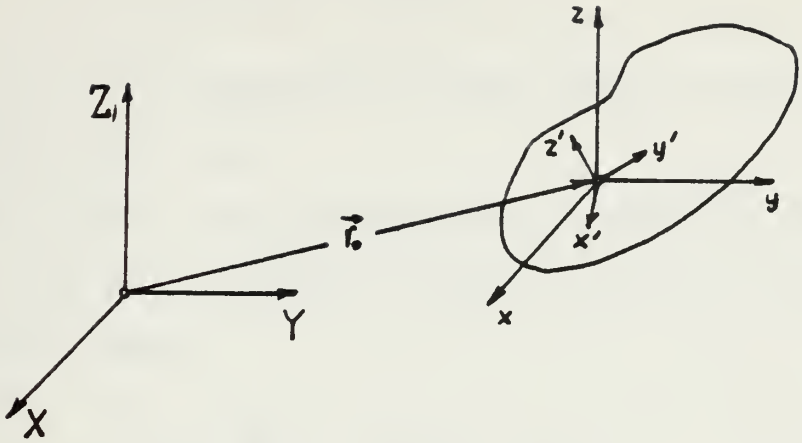


Figure 1-17

Rigid Body - Space Motion

The vector $d\vec{h}/dt$ is measured in the unprimed system, but it can be expressed as:

$$\frac{d\vec{h}}{dt} = \left(\frac{d\vec{h}}{dt}\right)' + \vec{\omega} \times \vec{h} \tag{1-270}$$

where $(d\vec{h}/dt)'$ is the same quantity measured in the primed system, and $\vec{\omega}$ is the angular velocity about the center of mass.

So Equation 1-269 is written in component form as:

$$\left(\frac{dh_i}{dt}\right)' + (\vec{\omega} \times \vec{h})_i = G_i \tag{1-271}$$

where $i$ refers to a primed axis now. Or better, using Equation 1-228

$$\sum_{j=1}^{3} \frac{d}{dt}(I_{ij}\,\omega_j)' + (\vec{\omega}\times\vec{h})_i = G_i \tag{1-272}$$

where $I_{ij}$ is a component of the inertia tensor calculated along the primed axes.

Notice that the primed axes unit vectors are vectors at rest, in the primed system, and obviously $I_{ij}$ is not changing with time. Hence,

$$\sum_{j=1}^{3} I_{ij}\,\frac{d\omega_j}{dt} + (\vec{\omega}\times\vec{h})_i = G_i \tag{1-273}$$

This result is interpreted as follows: Suppose the rigid body rotates about a principal axis with constant angular velocity $\vec{\omega}$. Rotation about a principal axis implies $\vec{\omega}$ parallel to $\vec{h}$, so that $\vec{\omega}\times\vec{h} = 0$. This simply means that the moment required for this rotation is zero.

This compares with the planar motion case result which comes by taking the time derivative of Equation 1-212.

$$\dot{h} = I\dot{\omega} = G \tag{1-274}$$

Equation 1-273 is the equation of rotational motion in the general case for rotation about a random axis, where external moment is required even for constant angular velocity.

It is now clear why we formulate the equations of motion in the primed, non-inertial coordinate system. We have avoided the time differentiation of the inertia tensor in Equation 1-272 by paying a little for the extra term $\vec{\omega}\times\vec{h}$.

In summary, we found the equations of motion of the rigid body described in the convenient form which will be used to derive the flight vehicle's equations of motion in Chapter 2. They are:

$$m\left(\frac{d\vec{V}}{dt} + \vec{\omega} \times \vec{V}\right) = \vec{F} \tag{1-275}$$

and

$$\sum_{j=1}^{3} I_{ij} \frac{d\omega_j}{dt} + (\vec{\omega} \times \vec{h})_i = G_i \tag{1-276}$$

where

$$h_i = \sum_{j=1}^{3} I_{ij}\, \omega_j \tag{1-277}$$

and $I_{ij}$ is calculated about the center of mass with respect to the primed axes.

Finally, if the primed axes are principal axes of the rigid body it is not difficult to see that the inertia tensor has only diagonal terms representing the principal moments of inertia (eigenvalues). This result comes from the diagonalization process of a square matrix. So, in this case Equation 1-276 is simplified as:

$$I_{ii} \frac{d\omega_i}{dt} + (\vec{\omega} \times \vec{h})_i = G_i \tag{1-278}$$

where

$$h_i = I_{ii}\, \omega_i \tag{1-279}$$

# CHAPTER 2

# VEHICLE'S EQUATIONS OF MOTION

## 2.1 INTRODUCTION

With the methods of obtaining rigid body equations of motion, established in the preceding chapter, we now proceed in this chapter to formulate the equations of motion in a most generally useful form.

The equations are basically simple in the sense that they are derived by making use of the basic Newtonian laws for forces and moments. Great advantages are obtained, as we have seen, by using a non-inertial frame of reference, because with an axis system fixed in the body, the inertial properties are constant and that simplifies the moment equations.

Some penalties are obtained by introducing non-inertial frames, the most important of which is the addition of extra terms in the basic equations of forces and moments, in trying to make the basic laws valid in this environment.

In any case, although the equations are simple, they are highly non-linear and not generally solvable mathematically. The formulation is greatly simplified by the use of linearization techniques, and small pertubation theory.

This method assumes an equilibrium position which is, in fact, a solution to these equations. That allows for the changes in forces and moments to be expressed as linear functions, in terms of small pertubations in linear, angular velocities, and other parameters involved.

The equations of motion will be derived for the body axis system established in Figure 1-11 and will be left in their dimentional form. Non-dimensional form will be mentioned as it applies to body axis system (Etkin's) and wind axis system (Perkin's).

Finally, the equations of motion will be split into two separate groups, the longitudinal and the lateral groups of equations.

## 2.2 FORCE EQUATIONS

Starting from the vector force equation

$$\sum \vec{F} = m\left(\frac{d\vec{V}}{dt} + \vec{\omega} \times \vec{V}\right) \tag{2-1}$$

where

$$\sum \vec{F} = \sum X \hat{i} + \sum Y \hat{j} + \sum Z \hat{k} \tag{2-2}$$

$$\vec{V} = u \hat{i} + v \hat{j} + w \hat{k} \tag{2-3}$$

$$\vec{\omega} = p \hat{i} + q \hat{j} + r \hat{k} \tag{2-4}$$

we can derive the force equations in component form.

The cross product term is expanded as:

$$\vec{\omega} \times \vec{V} = (qw - vr)\hat{i} + (-pw + ur)\hat{j} + (pv - qu)\hat{k} \tag{2-5}$$

and the force equations of motion are simply

$$\sum X = m(\dot{u} + qw - vr) \tag{2-6}$$

$$\sum Y = m(\dot{v} - pw + ur) \tag{2-7}$$

$$\sum Z = m(\dot{w} + pv - qu) \tag{2-8}$$

These equations describe the translational motion of the center of mass.

Consider the motion of the center of mass in the xz plane as Figure 2-1 shows, in two successive different locations on the flight path during time $\delta t$.

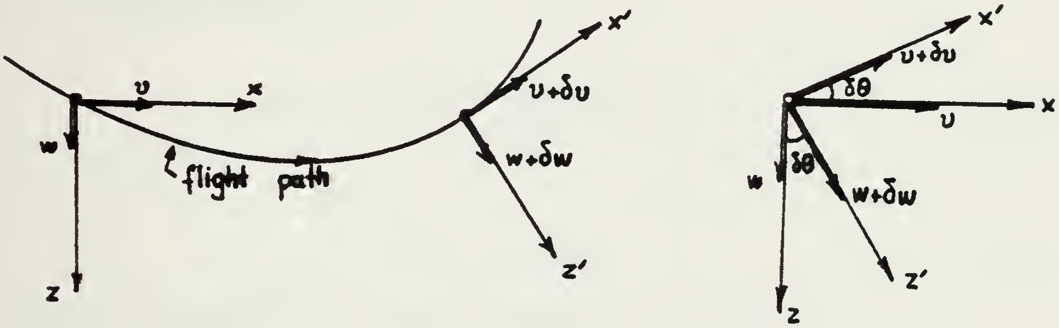


Figure 2-1

Motion of the Center of Mass in the xz-Plane

The acceleration along the x-axis is expressed as:

$$a_x = \frac{(u+\delta u)\cos\delta\theta + (w+\delta w)\sin\delta\theta - u}{\delta t} \tag{2-9}$$

For small pitch angle $\theta$, $\cos\delta\theta = 1$ and $\sin\delta\theta = \delta\theta$, so that

$$a_x = \frac{u+\delta u + w\delta\theta + \delta w\delta\theta - u}{\delta t} \tag{2-10}$$

letting now $\delta t \to 0$ and neglecting the higher order term $\delta w \delta \theta$, we obtain

$$a_x = \dot{u} + wq \tag{2-11}$$

and for the corresponding force equation

$$\Sigma X = m(\dot{u} + wq) \tag{2-12}$$

which is the same as Equation 2-6 by neglecting the term $vr$.

Similarly, the acceleration along the z-axis is expressed as:

$$a_z = \frac{(w+\delta w)\cos\delta\theta - (u+\delta u)\sin\delta\theta - w}{\delta t} \tag{2-13}$$

which is simplified as:

$$a_z = \frac{w+\delta w - u\delta\theta - \delta u\delta\theta - w}{\delta t} \tag{2-14}$$

or

$$a_z = \dot{w} - qu \tag{2-15}$$

and the corresponding force equation becomes:

$$\Sigma Z = m(\dot{w} - qu) \tag{2-16}$$

which is the same as Equation 2-8 by neglecting the term $pv$.

If we now consider the motion of the center of mass in the xy plane as Figure 2-2 shows, in two successive different locations on the flight path during time $\delta t$, we can derive the acceleration along the y-axis as:

$$a_y = \frac{(v+\delta v)\cos\delta\psi + (u+\delta u)\sin\delta\psi - v}{\delta t} \tag{2-17}$$

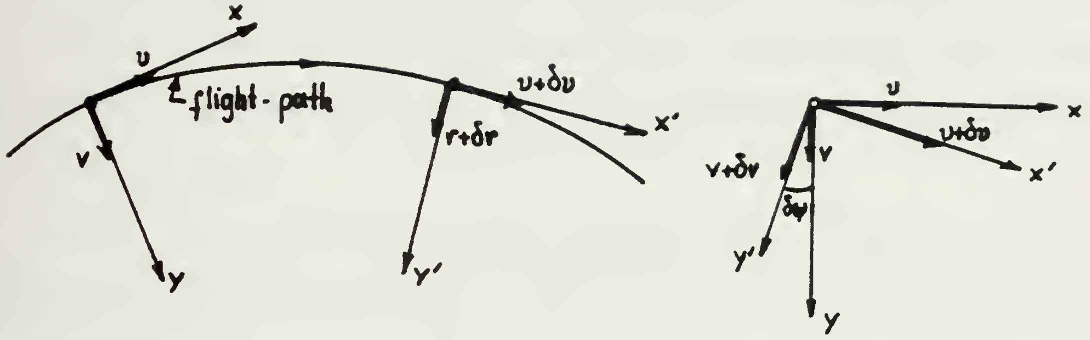


Figure 2-2

Motion of the Center of Mass in the xy-Plane

For small yaw angle $\psi$, $\cos\delta\psi=1$ and $\sin\delta\psi=\delta\psi$, so that

$$a_y = \frac{(v+\delta v + \upsilon\delta\psi + \delta\upsilon\delta\psi - v)}{\delta t} \tag{2-18}$$

letting now $\delta t \to 0$ and neglecting the higher order term $\delta\upsilon\delta\psi$, we obtain

$$a_y = \dot{v} + \upsilon r \tag{2-19}$$

and for the corresponding force equation

$$\Sigma Y = m(\dot{v} + \upsilon r) \tag{2-20}$$

which is the same as Equation 2-7 by neglecting the term $pw$.

In summary, the force equations are simplified neglecting higher order terms as:

$$\Sigma X = m(\dot{\upsilon} + qw) \tag{2-21}$$

$$\Sigma Y = m(\dot{v} + \upsilon r) \tag{2-22}$$

$$\Sigma Z = m(\dot{w} - q\upsilon) \tag{2-23}$$

## 2.3 VEHICLE SYMMETRIES

Up to this point we have made two assumptions. The first assumption was that earth is considered to be fixed in space so that we can establish an inertial reference frame on it. The second assumption was that the vehicle is considered to be a rigid body.

Facts are simplified further by considering the symmetries of the vehicle. For an airplane or a missile, the xz-plane is a plane of symmetry, if one assumes that the following holds for the density distribution.

$$\rho(x,y,z) = \rho(x,-y,z) \tag{2-24}$$

so that

$$I_{xy} = I_{yx} = -\iiint \rho(x,y,z)\, xy\, dx\,dy\,dz = -\iiint \rho(x,-y,z)\, xy\, dx\,dy\,dz \tag{2-25}$$

Notice that Equation 2-24 says that $\rho$ is an even function with respect to y but the integrant function is odd because it is a product of an even and odd function. Since the y integration has symmetric limits, the integral is zero.

For the same reason

$$I_{yz} = I_{zy} = -\iiint \rho(x,y,z)\, yz\, dx\,dy\,dz = 0 \tag{2-26}$$

The inertia tensor, therefore, for an airplane becomes:

$$\begin{vmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{vmatrix} \tag{2-27}$$

Notice also that the y axis is an eigenvector of the inertia tensor (Equation 2-27) because it satisfies the eigenvalue matrix equation:

$$\begin{vmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{vmatrix} \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} = I_{yy} \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} \qquad (2\text{-}28)$$

Hence, the y-axis is a principal axis, with $I_{yy}$ being the corresponding eigenvalue or principal moment of inertia. This was expected since y-axis is perpendicular to the plane of symmetry xz.

Unfortunately, this is the only symmetry that exists generally in any conventional aircraft.

In addition to this symmetry another symmetry can be found in gun projectiles and in most missiles and space-crafts. They have a second plane of symmetry, the xy plane, so that one can assume

$$\rho(x,y,z) = \rho(x,y,-z) \qquad (2\text{-}29)$$

and hence,

$$I_{xz} = I_{zx} = 0 \qquad (2\text{-}30)$$

Then, the inertia tensor for a two-symmetry vehicle becomes

$$\begin{vmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{vmatrix} \qquad (2\text{-}31)$$

Notice that in addition to the y-axis being a principal axis, the z-axis, as well as the x-axis, are all principal axes with corresponding principal moments of inertia $I_{zz}$, and $I_{xx}$, respectively.

## 2.4 MOMENT EQUATIONS

Starting from the vector moment equation:

$$\vec{G} = I \frac{d\vec{\omega}}{dt} + (\vec{\omega} \times \vec{h}) \qquad (2-32)$$

or in component form

$$G_i = \sum_{j=1}^{3} I_{ij} \frac{d\omega_j}{dt} + (\vec{\omega} \times \vec{h})_i \qquad (2-33)$$

where

$$h_i = I_{ij}\, \omega_j \qquad (2-34)$$

$$\vec{G} = L\,\hat{\imath} + M\,\hat{\jmath} + N\,\hat{k} \qquad (2-35)$$

$$\vec{h} = h_x\,\hat{\imath} + h_y\,\hat{\jmath} + h_z\,\hat{k} \qquad (2-36)$$

$$\frac{d\vec{\omega}}{dt} = \dot{p}\,\hat{\imath} + \dot{q}\,\hat{\jmath} + \dot{r}\,\hat{k} \qquad (2-37)$$

Equation 2-34 is expanded as:

$$h_x = p I_{xx} - r I_{xz} \qquad (2-38)$$

$$h_y = q I_{yy} \qquad (2-39)$$

$$h_z = -p I_{xz} + r I_{zz} \qquad (2-40)$$

also, the cross product terms is Equation 2-33 is expanded as

$$(\vec{\omega} \times \vec{h}) = (q h_z - r h_y)\,\hat{\imath} + (r h_x - p h_z)\,\hat{\jmath} + (p h_y - q h_x)\,\hat{k} \qquad (2-41)$$

or

$$
\begin{aligned}
(\vec{\omega}\times\vec{h}) = & \left(-qp\,I_{xz} + qr\,I_{zz} - rq\,I_{yy}\right)\hat{i} \\
& + \left(rp\,I_{xx} - r^2 I_{xz} + p^2 I_{xz} - rp\,I_{zz}\right)\hat{j} \\
& + \left(qp\,I_{yy} - qp\,I_{xx} + rq\,I_{xz}\right)\hat{k}
\end{aligned}
\tag{2-42}
$$

or

$$
\begin{aligned}
(\vec{\omega}\times\vec{h}) = & \left[-qp\,I_{xz} + qr\left(I_{zz} - I_{yy}\right)\right]\hat{i} \\
& + \left[rp\left(I_{xx} - I_{zz}\right) + I_{xz}\left(p^2 - r^2\right)\right]\hat{j} \\
& + \left[pq\left(I_{yy} - I_{xx}\right) + rq\,I_{xz}\right]\hat{k}
\end{aligned}
\tag{2-43}
$$

Thus, the moment equations are:

$$L = I_{xx}\,\dot{p} - I_{xz}\,\dot{r} - qp\,I_{xz} + qr\left(I_{zz} - I_{yy}\right) \tag{2-44}$$

$$M = I_{yy}\,\dot{q} + rp\left(I_{xx} - I_{zz}\right) + I_{xz}\left(p^2 - r^2\right) \tag{2-45}$$

$$N = I_{zz}\,\dot{r} - I_{xz}\,\dot{p} + pq\left(I_{yy} - I_{xx}\right) + rq\,I_{xz} \tag{2-46}$$

## 2.5 EXPANSION OF GRAVITY FORCES, THE LINEAR AND ANGULAR VELOCITY

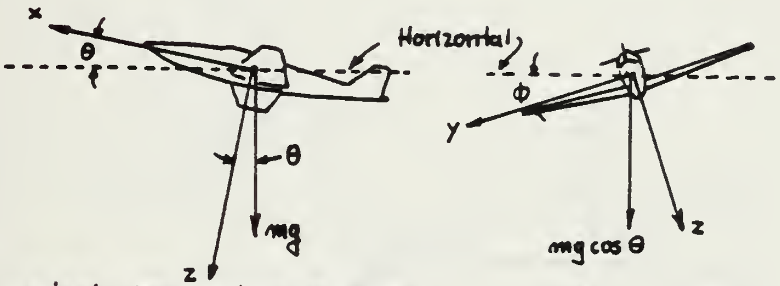
The total force acting on a flight vehicle can be split into a summation of aerodynamic forces (lift, drag and thrust) and a body force which is the gravity force.

Figure 2-3 shows an accidental orientation of the vehicle with respect to the vertical. The gravity force $\vec{W}$ can be expanded in components along the body axes as follows:

$$W_x = -mg \sin\Theta \qquad (2\text{-}47)$$

$$W_y = mg \cos\Theta \sin\phi \qquad (2\text{-}48)$$

$$W_z = mg \cos\Theta \cos\phi \qquad (2\text{-}49)$$



vertical plane containing x-axis and weight vector

view on yz plane

Figure 2-3

Expansion of the Gravity Force

For the expansion of the linear velocity vector along the body axes consider in Figure 2-4 the inertial reference frame being $Ox_1y_1z_1$. Transformation to the body axes coordinate system is obtrained by rotating the reference frame in an order-dependent manner as follows:

(a) Rotate around reference axis $Oz_1$ by the yaw angle $\psi$, which defines the intermediate reference system $Ox_2y_2z_2 = Ox_2y_2z_1$.

(b) Rotate around intermediate axis $Oy_2$ by the pitch angle $\theta$, which defines a second intermediate system $Ox_3y_3z_3 = Oxy_2z_3$.

(c) Rotate around final axis $Ox$ by the roll angle $\phi$, to define the desired body axes system $Oxyz = Ox_3yz$.

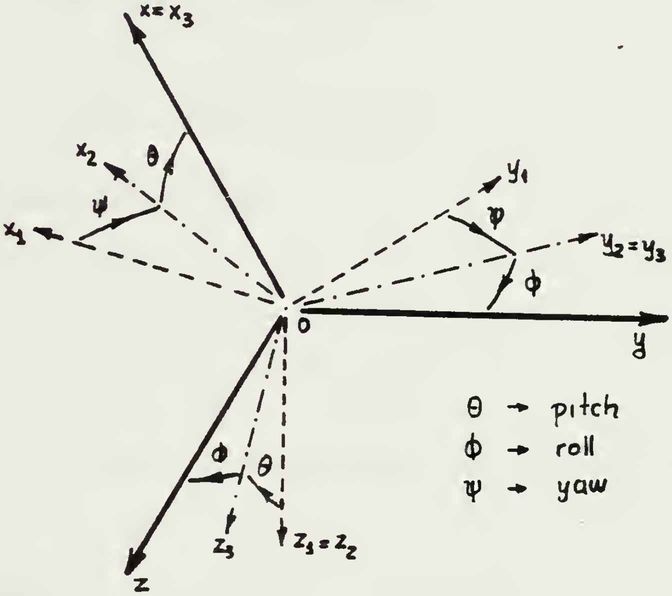


Figure 2-4

Expansion of the Linear Velocity

The components of the velocity vector in the body axes system (U,V,W) are defined in terms of the vector components in the fixed inertial frame ($U_1$, $V_1$, $W_1$) by means of the angles $\theta, \phi, \psi$ and the intermediate vector components ($U_2$, $V_2$, $W_2$) and ($U_3$, $V_3$, $W_3$).

The following relations hold:

for the first rotation

$$U_2 = U_1 \cos\psi + V_1 \sin\psi \tag{2-50}$$

$$V_2 = -U_1 \sin\psi + V_1 \cos\psi \tag{2-51}$$

$$W_2 = W_1 \tag{2-52}$$

for the second rotation

$$U_3 = U_2 \cos\theta - W_2 \sin\theta \tag{2-53}$$

$$V_3 = V_2 \tag{2-54}$$

$$W_3 = U_2 \sin\theta + W_2 \cos\theta \tag{2-55}$$

and for the third rotation

$$U = U_3 \tag{2-56}$$

$$V = V_3 \cos\phi + W_3 \sin\phi \tag{2-57}$$

$$W = -V_3 \sin\phi + W_3 \cos\phi \tag{2-58}$$

so that the velocity components along the body axes are expressed as

$$U = U_1 \cos\psi\cos\theta + V_1 \sin\psi\cos\theta - W_1 \sin\theta \tag{2-59}$$

$$V = U_1 (\cos\psi \sin\theta \sin\phi - \sin\psi \cos\phi) + V_1 (\sin\psi \sin\theta \sin\phi + \cos\psi \cos\phi) + W_1 \cos\theta \sin\psi \tag{2-60}$$

$$W = U_1 (\cos\psi \sin\theta \cos\phi + \sin\psi \sin\phi) + V_1 (\sin\psi \sin\theta \cos\phi - \cos\psi \sin\phi) + W_1 \cos\theta \cos\phi \tag{2-61}$$

Similar approach can lead us from the body axes system to the inertial for which

$$U_1 = U \cos\theta \cos\psi + V(\sin\phi \sin\theta \cos\psi - \cos\phi \sin\psi) + W(\cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi) \tag{2-62}$$

$$V_I = U\cos\theta\sin\psi + V(\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi) + W(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi) \qquad (2\text{-}63)$$

$$W_I = -U\sin\theta + V\sin\phi\cos\theta + W\cos\phi\cos\theta \qquad (2\text{-}64)$$

Integration of Equation 2-62, 2-63 and 2-64 give the expressions for the translational motion but in other than special simple cases they are not integrable.

Following a similar procedure we can expand the angular velocities p,q,r into components along the body axes.

With the vehicle in horizontal flight, one can see the yawing-velocity vector having the same direction as gravity, the rolling velocity vector having the direction of the longitudinal axis and the pitching velocity vector, perpendicular to both as shown in Figure 2-5. Those orientations are obtained according to right-hand rule.

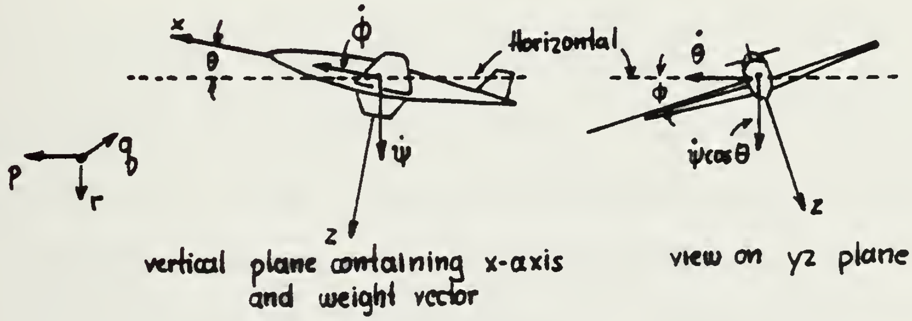


Figure 2-5

Expansion of the Angular Velocity

Then the angular velocity vector $\vec{\omega}$ can be expanded in the body axes as follows:

$$p = \dot{\phi} - \dot{\psi} \sin\theta \tag{2-65}$$

$$q = \dot{\theta} \cos\phi + \dot{\psi} \cos\theta \sin\phi \tag{2-66}$$

$$r = -\dot{\theta} \sin\phi + \dot{\psi} \cos\theta \cos\phi \tag{2-67}$$

One can now solve the Equations 2-65, 2-66, 2-67 for $\dot{\theta}$, $\dot{\psi}$, $\dot{\phi}$.

Multiplying Equation 2-66 by $\cos\phi$, Equation 2-67 by $\sin\phi$, and subtracting, we obtain:

$$\dot{\theta} = q \cos\phi - r \sin\phi \tag{2-68}$$

Multiplying Equation 2-66 by $\sin\phi$, Equation 2-67 by $\cos\phi$, and adding, we obtain

$$\dot{\psi} = q \frac{\sin\phi}{\cos\theta} + r \frac{\cos\phi}{\cos\theta} \tag{2-69}$$

Then substituting Equation 2-69 into Equation 2-65, we obtain

$$\dot{\phi} = p + q \sin\phi \tan\theta + r \cos\phi \tan\theta \tag{2-70}$$

Integration of Equations 2-68, 2-69 and 2-70 give the expressions for the rotational motion but in other than special simple cases they are not integrable.

We can now modify the vehicle's equations of motion by considering the gravity as follows:

Force equations:

$$X = m ( \dot{u} + qw - vr + g \sin\theta ) \tag{2-71}$$

$$Y = m ( \dot{v} + ur - pw - g \cos\theta \sin\phi ) \tag{2-72}$$

$$Z = m ( \dot{w} + pv - qu - g \cos\theta \cos\phi ) \tag{2-73}$$

where now the X,Y,Z forces represent only aerodynamic forces (i.e., contributions of lift, drag and thrust).

Moment equations:

$$L = I_{xx}\dot{p} - I_{xz}\dot{r} - qpI_{xz} + qr(I_{zz} - I_{yy}) \quad (2\text{-}74)$$

$$M = I_{yy}\dot{q} + rp(I_{xx} - I_{zz}) + I_{xz}(p^2 - r^2) \quad (2\text{-}75)$$

$$N = I_{zz}\dot{r} - I_{xz}\dot{p} + pq(I_{yy} - I_{xx}) + rqI_{xz} \quad (2\text{-}76)$$

with auxiliary relations:

$$p = \dot{\phi} - \dot{\psi}\sin\theta \quad (2\text{-}77)$$

$$q = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \quad (2\text{-}78)$$

$$r = -\dot{\theta}\sin\phi + \dot{\psi}\cos\theta\cos\phi \quad (2\text{-}79)$$

Notice that the gravity force contributes zero moments because it acts at the center of mass.

## 2.6 LINEARIZATION OF FLIGHT VEHICLE'S EQUATIONS

The three force and the threee moment equations derived in Section 2.5 must be linearized for use in the stability and control analysis. It is assumed that the motion of the vehicle consists of small deviations about a reference steady flight path.

A steady flight path is described by the general equations of motion, if we consider all time derivatives zero. We denote by capital letters and the subscript (o), the steady flight conditions, obtaining thus:

$$X_o = m(Q_oW_o - V_oR_o + g\sin\Theta_o) \quad (2\text{-}80)$$

$$Y_o = m(U_oR_o - P_oW_o - g\cos\Theta_o\sin\Phi_o) \quad (2\text{-}81)$$

$$Z_o = m(P_oV_o - Q_oU_o - g\cos\Theta_o\cos\Phi_o) \quad (2\text{-}82)$$

$$L_0 = -Q_0P_0 I_{xz} + Q_0R_0 (I_{zz}-I_{yy}) \tag{2-83}$$

$$M_0 = R_0P_0 (I_{xx}-I_{zz}) + I_{xz} (P_0^2 - R_0^2) \tag{2-84}$$

$$N_0 = P_0Q_0 (I_{yy}-I_{xx}) + R_0Q_0 I_{xz} \tag{2-85}$$

By subtracting from the original equations the above steady flight equations, we obtain the perturbed equations. For example, subtracting Equation 2-80 from Equation 2-71 we get:

$$X-X_0 = m\left[\dot{u} + qw - Q_0W_0 - vr + V_0R_0 + g(\sin\theta - \sin\theta_0)\right] \tag{2-86}$$

Notice that

$$qw = (Q_0+\delta q)(W_0+\delta w) = Q_0W_0 + W_0\,\delta q + Q_0\,\delta w + \cancelto{\text{H.O.T.}}{\delta q\,\delta w} \tag{2-87}$$

Similarly

$$vr = V_0R_0 + V_0\,\delta r + R_0\,\delta v \tag{2-88}$$

Also

$$\sin\theta = \sin(\theta_0+\delta\theta) = \sin\theta_0\cos\delta\theta + \cos\theta_0\sin\delta\theta \tag{2-89}$$

For small perturbation quantities, Equation 2-89 becomes:

$$\sin\theta = \sin\theta_0 + (\cos\theta_0)\,\delta\theta \tag{2-90}$$

Combining all these, Equation 2-86 becomes

$$\delta X = m\left[\dot{u} + W_0\delta q + Q_0\delta w - V_0\,\delta r - R_0\,\delta v + (g\cos\theta_0)\,\delta\theta\right] \tag{2-91}$$

Although we are deriving perturbed equations, it is more convenient to drop the differential symbol $\delta$.

Similar work leads us then to the following set of linearized equations:

$$X = m[\dot{u} + W_0 q + Q_0 w - V_0 r - R_0 v + (g\cos\Theta_0)\theta] \quad (2\text{-}92)$$

$$Y = m[\dot{v} + U_0 r + R_0 u - W_0 p - P_0 w - (g\cos\Theta_0\cos\Phi_0)\phi + (g\sin\Theta_0\sin\Phi_0)\theta] \quad (2\text{-}93)$$

$$Z = m[\dot{w} + V_0 p + P_0 u - U_0 q - Q_0 u + (g\cos\Theta_0\cos\Phi_0)\phi + (g\sin\Theta_0\cos\Phi_0)\theta] \quad (2\text{-}94)$$

$$L = I_{xx}\dot{p} - I_{xz}\dot{r} - (Q_0 p + P_0 q)I_{xz} + (Q_0 r + R_0 q)(I_{zz} - I_{yy}) \quad (2\text{-}95)$$

$$M = I_{yy}\dot{q} + (R_0 p + P_0 r)(I_{xx} - I_{zz}) + I_{xz}(2P_0 p - 2R_0 r) \quad (2\text{-}96)$$

$$N = I_{zz}\dot{r} - I_{xz}\dot{p} + (P_0 q + Q_0 p)(I_{yy} - I_{xx}) + (R_0 q + Q_0 r)I_{xz} \quad (2\text{-}97)$$

In addition, for the auxiliary relations described by Equations 2-77, 2-78, 2-79, the corresponding perturbed equations are:

$$p = \dot{\phi} - \dot{\psi}\sin\Theta_0 - (\dot{\Psi}_0\cos\Theta_0)\theta \quad (2\text{-}98)$$

$$q = \dot{\theta}\cos\Phi_0 - (\dot{\Psi}_0\sin\Theta_0\sin\Phi_0)\theta + (\dot{\Psi}_0\cos\Theta_0\cos\Phi_0 - \dot{\Theta}_0\sin\Phi_0)\phi + \dot{\psi}\cos\Theta_0\sin\Phi_0 \quad (2\text{-}99)$$

$$r = \dot{\psi}\cos\Theta_0\cos\Phi_0 - (\dot{\Psi}_0\cos\Theta_0\sin\Phi_0 + \dot{\Theta}_0\cos\Phi_0)\phi - \dot{\theta}\sin\Phi_0 - (\dot{\Psi}_0\sin\Theta_0\cos\Phi_0)\theta \quad (2\text{-}100)$$

Although we have obtained linear equations, they are extremely complex and not suitable for control analysis.

A further simplification is obtained if we consider the reference flight path to be straight, i.e.

$$\dot{\Psi}_0 = \dot{\Theta}_0 = 0 \quad (2\text{-}101)$$

with no angular velocities

$$P_0 = Q_0 = R_0 = 0 \quad (2\text{-}102)$$

wings level flight, i.e.

$$\Phi_0 = 0 \quad (2\text{-}103)$$

symmetric, i.e.

$$\Psi_0 = 0 \tag{2-104}$$

$$V_0 = 0 \tag{2-105}$$

With this assumption, the reference flight path is described by the equations:

$$X_0 = mg \sin \Theta_0 \tag{2-106}$$

$$Y_0 = 0 \tag{2-107}$$

$$Z_0 = -mg \cos \Theta_0 \tag{2-108}$$

$$L_0 = M_0 = N_0 = 0 \tag{2-109}$$

while the perturbed equations become

$$X = m \left[ \dot{u} + W_0 q + g \cos \Theta_0 \, \theta \right] \tag{2-110}$$

$$Y = m \left[ \dot{v} + U_0 r + W_0 p - g \cos \Theta_0 \, \phi \right] \tag{2-111}$$

$$Z = m \left[ \dot{w} - U_0 q + g \sin \Theta_0 \, \theta \right] \tag{2-112}$$

$$L = I_{xx} \dot{p} - I_{xz} \dot{r} \tag{2-113}$$

$$M = I_{yy} \dot{q} \tag{2-114}$$

$$N = I_{zz} \dot{r} - I_{xz} \dot{p} \tag{2-115}$$

with auxilliary relations:

$$p = \dot{\phi} - \dot{\psi} \sin \Theta_0 \tag{2-116}$$

$$q = \dot{\theta} \tag{2-117}$$

$$r = \dot{\psi} \cos \Theta_0 \tag{2-118}$$

We find that the equations are separable into two groups. The first group associates the equations that describe the motion in the xz body plane, called longitudinal equations and the second group associates the equations that describe the motion in the xy body plane, called lateral directional equations.

We find also, the force equations divided by mass and the moment equations divided by the corresponding moment of inertia.

In the longitudinal equations the independent variables are:

$\upsilon$ = x-axis linear velocity

w = z-axis linear velocity

q = pitching angular velocity

In the lateral-directional equations the independent variables are:

v = y-axis linear velocity

p = rolling angular velocity

r = yawing angular velocity

with all these quantities being perturbations from the initial flight conditions.

We arrive then to the following form of equations:

Longitudinal equations

$$\frac{1}{m} X = \dot{\upsilon} + W_0 q + g\cos\Theta_0\,\Theta \qquad (2\text{-}119)$$

$$\frac{1}{m} Z = \dot{w} - U_0 q + g\sin\Theta_0\,\Theta \qquad (2\text{-}120)$$

$$\frac{1}{I_{yy}} M = \dot{q} \qquad (2\text{-}121)$$

Lateral-directional equations

$$\frac{1}{m} Y = \dot{v} + U_0 r - W_0 p - g\cos\Theta_0\,\phi \qquad (2\text{-}122)$$

$$\frac{1}{I_{xx}} L = \dot{p} - \frac{I_{xz}}{I_{xx}} \dot{r} \tag{2-123}$$

$$\frac{1}{I_{zz}} N = \dot{r} - \frac{I_{xz}}{I_{zz}} \dot{p} \tag{2-124}$$

For further simplification, it is convenient to introduce here the wind axes system, which as we have seen is produced by a rotation about the y-body axis so that the x-wind axis points in the opposite direction of the relative wind. This axis is always tangent to the flight path and makes an angle of flight path $\gamma_0$ with the horizontal, as shown in Figure 2-6.

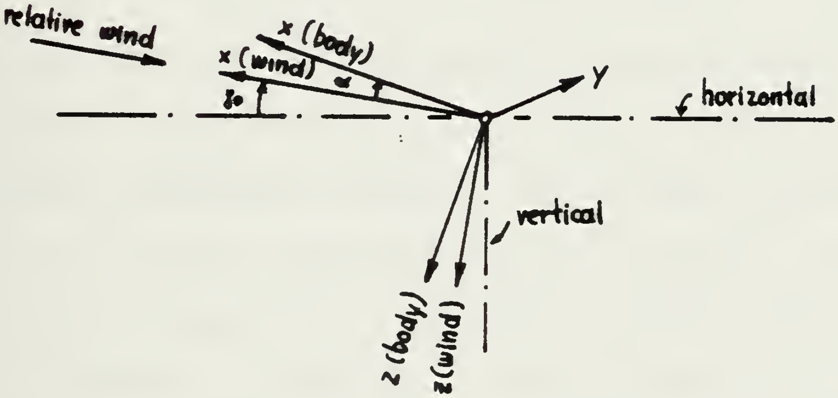


Figure 2-6

Direction of the Body Axes with Respect to Wind Axes in Straight Level Perturbed Condition

Since

$$\theta = \gamma + \alpha \tag{2-125}$$

in the steady situation

$$\theta_o = \gamma_o + \alpha_o \tag{2-126}$$

where

$$\alpha_o = 0 \tag{2-127}$$

In fact, in order to maintain straight level flight, $\alpha_o$ has to be small. The appropriate relation for expressing the angle of attack $\alpha$ in terms of the w-velocity if obtained by Equation 1-194 if one considers small $\alpha$ and $w$, so that one can express:

$$\alpha = w/U_o \tag{2-128}$$

From Equation 2-127 then we deduce:

$$\alpha_o \cong W_o/U_o \cong 0 \qquad \Rightarrow \quad W_o \cong 0 \tag{2-129}$$

Notice that by this alignment, since the wind axes are fixed for fixed relative wind, we can measure the perturbation quantities about $\gamma_o$ in the wind frame.

It is further desirable to express the Lateral Directional equations in terms of the sideship angle $\beta$ instead of the v- velocity. The appropriate relation for expressing the angle $\beta$ in terms of the v- velocity is obtained by Equation 1-195, if one considers small $\beta$ and $v$, so that one can express

$$\beta = v/U_o \tag{2-130}$$

So introducing

from Equations 2-126 and 2-127

$$\theta_o = \gamma_o \tag{2-131}$$

from Equation 2-129

$$W_o = 0 \tag{2-132}$$

from Equation 2-130

$$v = \beta U_o \tag{2-133}$$

we obtain

Longitudinal Equations

$$\begin{aligned} \frac{1}{m} X &= \dot{u} + g \cos\gamma_o \Theta \\ \frac{1}{m} Z &= \dot{w} - U_o q + g \sin\gamma_o \Theta \\ \frac{1}{I_{yy}} M &= \dot{q} \end{aligned} \tag{2-134}$$

Lateral Directional Equations

$$\begin{aligned} \frac{1}{m} Y &= \dot{v} + U_o r - g \cos\gamma_o \phi \\ \frac{1}{I_{xx}} L &= \dot{p} - \frac{I_{xz}}{I_{xx}} \dot{r} \\ \frac{1}{I_{zz}} N &= \dot{r} - \frac{I_{xz}}{I_{zz}} \dot{p} \end{aligned} \tag{2-135}$$

with auxilliary relations:

$$\begin{aligned} p &= \dot{\phi} - \dot{\psi} \sin\gamma_o \\ q &= \dot{\Theta} \\ r &= \dot{\psi} \cos\gamma_o \end{aligned} \tag{2-136}$$

The only thing that remains is to express the left-hand side forces and moments in series of terms involving aerodynamic derivatives and small perturbation quantities, that in fact generate the forces and moments.

## 2.7 EXPANSION OF FORCES AND MOMENTS

In Section 2.6, it was assumed that the motion of the vehicle consists of small deviations around a reference steady flight path.

Small disturbance theory applied on the perturbed equations of motion assumes that the aerodynamic forces and moments are functions of the disturbance velocities (angular and linear), the control surface deflections and their derivatives.

In general, for a function

$$K(\lambda, \mu, \nu, \ldots) \tag{2-137}$$

the Taylor series expansion about a reference value

$$K_o(\lambda_o, \mu_o, \nu_o, \ldots) \tag{2-138}$$

assuming that the function has derivatives of all orders defined, is written as

$$K = K_o + \delta K = K_o + \left(\frac{\partial K}{\partial \lambda}\right)_o \delta\lambda + \left(\frac{\partial K}{\partial \mu}\right)_o \delta\mu + \left(\frac{\partial K}{\partial \nu}\right)_o \delta\nu + \frac{1}{2!}\left[\left(\frac{\partial^2 K}{\partial \lambda^2}\right)_o (\delta\lambda)^2 + 2\left(\frac{\partial^2 K}{\partial \lambda \partial \mu}\right)_o \delta\lambda\,\delta\mu + \left(\frac{\partial^2 K}{\partial \mu^2}\right)_o (\delta\mu)^2\right] + \cdots\cdots \tag{2-139}$$

where the derivatives must be evaluated at the reference value denoted by the subscript (o).

Now, making the assumption that the deviations $\delta\lambda$, $\delta\mu$, $\delta\nu$, are so small that product terms are essentially zero, we obtain

$$\delta K = \left(\frac{\partial K}{\partial \lambda}\right)_o \delta\lambda + \left(\frac{\partial K}{\partial \mu}\right)_o \delta\mu + \left(\frac{\partial K}{\partial \nu}\right)_o \delta\nu + \cdots\cdots \tag{2-140}$$

Using this method and provided that the above assumption is valid, we succeed to express the perturbation quantity $\delta K$ as a linear combination of the perturbation parameters from which it depends upon.

People in the field like to denote the partial derivatives as $K_\lambda$, $K_\mu$, $K_\nu$, etc., instead of $(\partial K/\partial\lambda)_o$, $(\partial K/\partial\mu)_o$, $(\partial K/\partial\nu)_o$, etc. In addition, according to the convention met in Section 2.6 of dropping the $\delta$, we obtain:

$$K = K_\lambda \lambda + K_\mu \mu + K_\nu \nu + \cdots\cdots \tag{2-141}$$

where $K_\lambda$, $K_\mu$, $K_\nu$, are called stability derivatives.

Fortunately, not all the stability derivatives in an expansion of a force or a moment have a value. Some of them are zero or essentially zero.

Expansion then of forces and moments, in terms of the significant stability derivatives yields:

For longitudinal equations:

$$\frac{1}{m} X = X_u u + X_w w + X_\delta \delta \tag{2-142}$$

$$\frac{1}{m} Z = Z_u u + Z_w w + Z_{\dot{w}} \dot{w} + Z_q q + Z_\delta \delta \tag{2-143}$$

$$\frac{1}{I_{yy}} M = M_u u + M_w w + M_{\dot{w}} \dot{w} + M_q q + M_\delta \delta \tag{2-144}$$

where it is clear that

$$X_u = \frac{1}{m} \cdot \frac{\partial X}{\partial u} \qquad \text{e.t.c.} \tag{2-145}$$

and denotes the control input due to control surfaces deflections.

For the lateral-directional equations:

$$\frac{1}{m} Y = Y_v v + Y_{\dot v} \dot v + Y_p p + Y_r r + Y_\delta \delta \tag{2-146}$$

$$\frac{1}{I_{xx}} L = L_\beta \beta + L_{\dot\beta} \dot\beta + L_p p + L_r r + L_\delta \delta \tag{2-147}$$

$$\frac{1}{I_{zz}} N = N_\beta \beta + N_{\dot\beta} \dot\beta + N_p p + N_r r + N_\delta \delta \tag{2-148}$$

where it is clear that

$$Y_v = \frac{1}{m}\frac{\partial Y}{\partial v} \quad , \quad Y_p = \frac{1}{m}\frac{\partial Y}{\partial p} \tag{2-149}$$

Therefore the equations of motion become:

Longitudinal Equations

$$\begin{aligned} &\dot u + X_u u - X_w w + g\cos\gamma_0 \theta = X_\delta \delta \\ &-Z_u u + \dot w - Z_w w - Z_{\dot w} \dot w - U_0 q - Z_q q + g \sin\gamma_0 \theta = Z_\delta \delta \\ &-M_u u - M_w w - M_{\dot w} \dot w + \dot q - M_q q = M_\delta \delta \end{aligned} \tag{2-150}$$

Lateral Directional Equations

$$\begin{aligned} &\dot v - Y_v v - Y_{\dot v} \dot v - Y_p p + U_0 r - Y_r r - g\cos\gamma_0 \phi = Y_\delta \delta \\ &-L_\beta \beta - L_{\dot\beta} \dot\beta + \dot p - L_p p - I_{xz}/I_{xx}\, \dot r - L_r r = L_\delta \delta \\ &-N_\beta \beta - N_{\dot\beta} \dot\beta - I_{xz}/I_{xx}\, \dot p - N_p p + \dot r - N_r r = N_\delta \delta \end{aligned} \tag{2-151}$$

with auxilliary relations of Equation 2-136.

It is more convenient to Laplace Transform the equations and arrange them in matrix form. Further augmenting the

equations by the auxilliary relations, using Equation 2-130 for the lateral directional set and dividing the Y-equation by $U_0$ we obtain the form

Longitudinal Equations

$$\begin{vmatrix} s-X_u & -X_w & 0 & g\cos\gamma_0 \\ -Z_u & s-sZ_{\dot{w}}-Z_w & -(U_0+Z_q) & g\sin\gamma_0 \\ -M_u & -sM_{\dot{w}}-M_w & s-M_q & 0 \\ 0 & 0 & 1 & -s \end{vmatrix} \cdot \begin{vmatrix} u \\ w \\ q \\ \theta \end{vmatrix} = \begin{vmatrix} X_\delta \\ Z_\delta \\ M_\delta \\ 0 \end{vmatrix} \delta \qquad (2\text{-}152)$$

Lateral-Directional Equations

$$\begin{vmatrix} s-sY_{\dot{v}}-Y_v & -Y_p^* & 1-Y_r^* & -\frac{g}{U_0}\cos\gamma_0 & 0 \\ -sL_{\dot{\beta}}-L_\beta & s-L_p & -s\frac{I_{xz}}{I_{xx}}-L_r & 0 & 0 \\ -sN_{\dot{\beta}}-N_\beta & -s\frac{I_{xz}}{I_{xx}}-N_p & s-N_r & 0 & 0 \\ 0 & 1 & 0 & -s & s(\sin\gamma_0) \\ 0 & 0 & 1 & 0 & -s(\cos\gamma_0) \end{vmatrix} \begin{vmatrix} \beta \\ p \\ r \\ \phi \\ \psi \end{vmatrix} = \begin{vmatrix} Y_\delta^* \\ L_\delta \\ N_\delta \\ 0 \\ 0 \end{vmatrix} \delta \qquad (2\text{-}153)$$

where

$$Y_p^* = Y_p / U_0 \qquad (2\text{-}154)$$

$$Y_r^* = Y_r / U_0 \qquad (2\text{-}155)$$

$$Y_\delta^* = Y_\delta / U_0 \qquad (2\text{-}156)$$

We will further simplify the equations of motion in the following chapters in a form that is more suitable for obtaining aerodynamic transfer functions, and computer solutions.

Our interest first will be focused on examining the aerodynamic stability derivatives. They will be left in their dimensional form rather than non-dimensional.

## CHAPTER 3

## NON-DIMENSIONAL STABILITY DERIVATIVES

### 3.1 INTRODUCTION

Stability is a very important concept in successful vehicle flight. We have encountered the terms "stability" and "equilibrium" but we did not emphasize particularly those concepts.

In the last chapter we mathematically obtained equations of motion and linearized them about a referenced straight flight which was assumed to be an equilibrium condition.

But what is equilibrium? Equilibrium is a state of rest or uniform (unaccelerated motion) in a straight line where all forces and moments acting on a body are balanced out.

Stability is a property of an equilibrium state. We will consider two types of stability which are of great interest in flight vehicle dynamics, defined as follows: Static stability is defined by the direction and magnitude of the initial tendency to return to equilibrium condition after a small disturbance has occurred. It is distinguished by positive static stability if the body tends to return to its equilibrium state, negative static stability if the body diverges from its equilibrium state, and neutral static stability if the body remains in the disturbed position.

As an example, the pendulum considered in Section 1.3 has a positive static stability at its $\theta=0^\circ$ position but has a negative static stability at its $\theta=180^\circ$ position, while its motion remains invariant if the pivot is moved in any other position and is said to have a neutral static stability.

Dynamic stability determines the resulting motion in time of the body if initially disturbed. It is distinguished similarly in positive, negative, and neutral dynamic stability.

Various conditions of dynamic behavior were considered in Chapter 1, in analyzing spring mass damper systems. Figure 3-1 summarizes all possible time responses of any dynamical system.

Dynamic stability implies the existence of static stability, but the opposite is not necessarily true.

Now the geometry of the vehicle comes to play a unique and important role in specifying any particular mode of dynamical motion. All surfaces exposed to the airstream contribute forces and moments about the center of gravity, which are expressed as we have seen in the last chapter as linear combinations of stability derivatives.

The physical meaning of each of the stability derivatives is straight forward. Thus, $X_\upsilon$ is the rate of chance of the force generated in the x-direction due to a $\upsilon$-velocity perturbation, divided by the mass of the vehicle, while $X_\upsilon \upsilon$

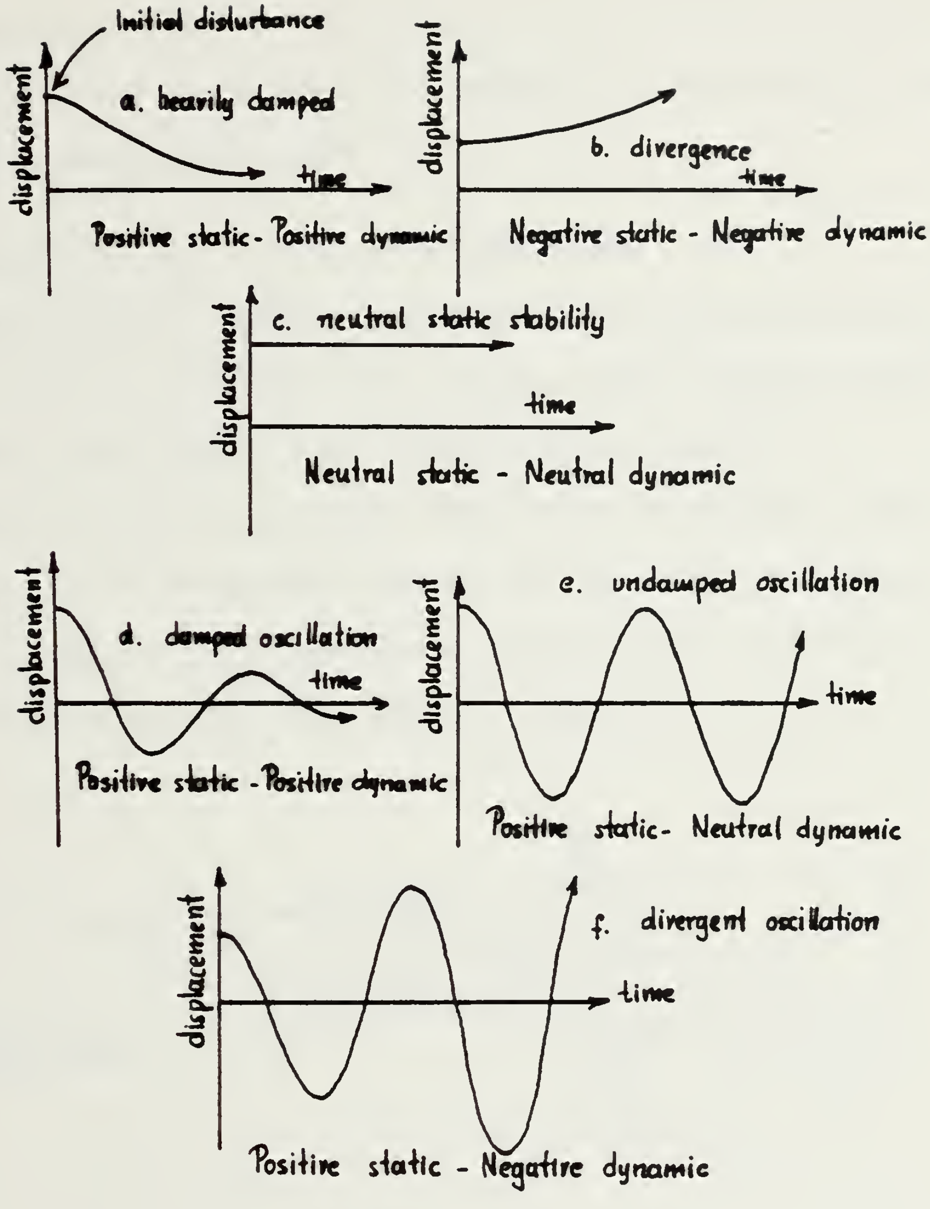

Initial disturbance
displacement
a. heavily damped
time
Positive static - Positive dynamic
displacement
b. divergence
time
Negative static - Negative dynamic
displacement
c. neutral static stability
time
Neutral static - Neutral dynamic
displacement
d. damped oscillation
time
Positive static - Positive dynamic
displacement
e. undamped oscillation
time
Positive static - Neutral dynamic
displacement
f. divergent oscillation
time
Positive static - Negative dynamic

Figure 3-1
Various Responses of a Dynamical System to a Small Disturbance
a,b,c Non-Oscillatory Modes, d,e,f Oscillatory Modes

is the force generated in the x-direction due to a $v$-velocity perturbation divided by the mass of the vehicle.

In this chapter we will give more insight in evaluating the stability derivatives in terms of the vehicle's geometry and airstream properties.

## 3.2 LONGITUDINAL STABILITY DERIVATIVES

Longitudinal stability derivatives are encountered in the longitudinal equations and, hance, they uniquely specify the X-force, the Z-force and the M-pitching moment.

Dimensional analysis approach indicates that we can express the aerodynamic forces and moments in terms of their corresponding non-dimensional coefficients which are usually deduced from wind tunnel tests of models.

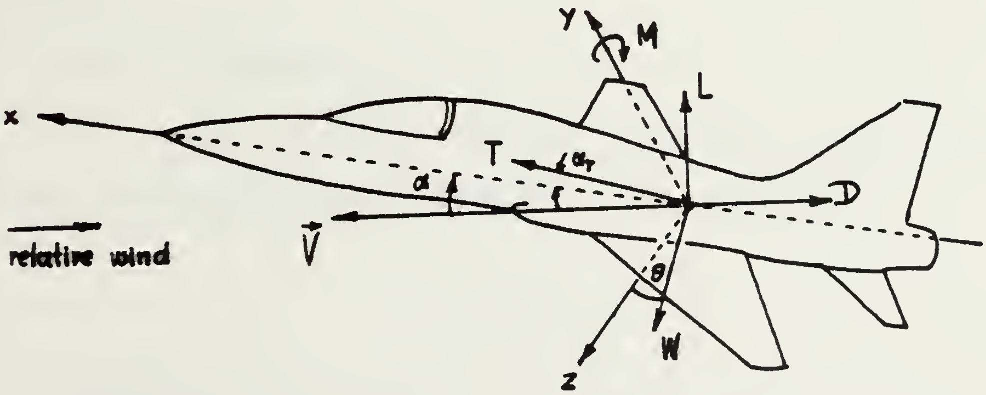


Figure 3-2

Forces and Moments Considered in Longitudinal Motion

In particular, we can express the longitudinal forces and moments as

lift $L = C_L \frac{1}{2} \rho V^2 S$ (3-1)

drag $D = C_D \frac{1}{2} \rho V^2 S$ (3-2)

pitching moment $M = C_M \frac{1}{2} \rho V^2 S c$ (3-3)

where $C_L, C_D, C_M$ are the corresponding lift, drag, and pitching moment coefficients, $\rho$ the air density, V the velocity of the vehicle, S is a reference area usually taken the wing area, and c, a reference length, usually taken the length of the mean aerodynamic chord.

The Longitudinal Stability Derivatives can be evaluated as follows:

1. The $X_\upsilon$ Derivative

Is the rate of change in the X-force due to a $\upsilon$-velocity perturbation divided by the mass m.

$$X_\upsilon = \frac{1}{m} \frac{\partial X}{\partial \upsilon} \tag{3-4}$$

According to Figure 3-2, the X-force is expressed as:

$$X = L \sin\alpha - D \cos\alpha + T \cos\alpha_T \tag{3-5}$$

where T denotes the thrust.

For $\alpha$ small, $\alpha_T$ essentially zero, and making use of Equation 3-2 we obtain:

$$X = -\frac{1}{2}\rho V^2 S C_D + T \qquad (3\text{-}6)$$

Notice that because a small change only in $u$-velocity occurs

$$V^2 = (U_o + u)^2 + v^2 + w^2 \qquad (3\text{-}7)$$

and since products of perturbations are essentially zero

$$V^2 = U_o^2 + 2U_o u \qquad (3\text{-}8)$$

where $U_o$ is the reference flight forward velocity which for practical purposes is assumed equal to V.

Then Equation 3-6 becomes

$$X = -\frac{1}{2}\rho\left(U_o^2 + 2U_o u\right) S C_D + T \qquad (3\text{-}9)$$

Neglecting the effect of the $u$-velocity on the thrust T, by differentiating Equation 3-9 with respect to $u$, and dividing by the mass m, we obtain

$$X_u = -\frac{\rho U_o S}{m} C_D - \frac{1}{2}\,\frac{\rho V^2 S}{m}\cdot\frac{\partial C_D}{\partial u} \qquad (3\text{-}10)$$

Or since $V \doteq U_o$

$$X_u = -\frac{\rho U_o S}{m}\left(C_D + \frac{U_o}{2}\cdot\frac{\partial C_D}{\partial u}\right) \qquad (3\text{-}11)$$

Finally, if we define

$$C_{D_u} = \frac{U_o}{2}\cdot\frac{\partial C_D}{\partial u} \qquad (3\text{-}12)$$

we arrive at the following final form of the $X_u$ derivative

$$X_u = -\frac{\rho S U_o}{m}\left(C_D + C_{D_u}\right) \qquad (3\text{-}13)$$

The quantity $C_{D_\upsilon}$ is essentially zero for low subsonic speed, but sometimes near the critical mach number ($0.8<M<1.0$) where a large increase in drag occurs, $C_{D_\upsilon}$ obtains a large positive value.

### 2. The $Z_\upsilon$ Derivative

It is the rate of change in the Z-force, due to a $\upsilon$-velocity perturbation divided by the mass m.

$$Z_\upsilon = \frac{1}{m}\frac{\partial Z}{\partial \upsilon} \tag{3-14}$$

According to Figure 3-2, the Z-force is expressed as:

$$Z = W\cos\theta - L\cos\alpha - D\sin\alpha - T\sin\alpha_T \tag{3-15}$$

which is simplified to:

$$Z = W\cos\theta - L \tag{3-16}$$

Neither W or $\theta$ depend on $\upsilon$-velocity. Since by similar approach as in the $X_\upsilon$ derivative, we obtain:

$$Z_\upsilon = -\frac{\rho S \upsilon_0}{m}\left(C_L + C_{L_\upsilon}\right) \tag{3-17}$$

where

$$C_{L_\upsilon} = \frac{\upsilon_0}{2}\cdot\frac{\partial C_L}{\partial \upsilon} \tag{3-18}$$

The quantity $C_{L_\upsilon}$ is essentially zero for low subsonic speed, but sometimes near the critical mach number may reach a considerable value and its sign can be changed depending on the airframe geometry, mach number, dynamic pressure and aeroelastic properties.

For subsonic straight level flight ($\gamma_0=0$)

$$C_L = \frac{W}{\rho U_0^2 S/2} \qquad (3\text{-}19)$$

and $Z_u$ is expressed simply as

$$Z_u = -\frac{2g}{U_0} \qquad (3\text{-}20)$$

### 3. The $M_u$ Derivative

It is the rate of change in the pitching moment M due to a $u$-velocity perturbation, divided by the $I_{yy}$ moment of inertia.

$$M_u = \frac{1}{I_{yy}} \frac{\partial M}{\partial u} \qquad (3\text{-}21)$$

The $M_u$ derivative is evaluated simply by differentiating Equation 3-3 with respect to $u$.

$$M_u = \frac{\rho S U_0 c}{I_{yy}} \left( C_M + C_{M_u} \right) \qquad (3\text{-}22)$$

where

$$C_{M_u} = \frac{U_0}{2} \cdot \frac{\partial C_M}{\partial u} \qquad (3\text{-}23)$$

The quantity $C_{M_u}$ is very sensitive upon the variation of the $u$-velocity and its sign can change depending upon the geometry, aeroelastic properties, mach number and dynamic pressure. Thrust can also affect $C_{M_u}$ for propeller driven aircraft.

### 4. The $X_w$ Derivative

It is the rate of change in X-force, due to a w-velocity perturbation divided by the mass m.

$$X_w = \frac{1}{m}\frac{\partial X}{\partial w} \tag{3-24}$$

A w-velocity perturbation causes a small change in the angle of attack $\alpha$. Thus, in this case we cannot neglect the angle of attack $\alpha$ which is approximately given by:

$$\alpha \cong \frac{w}{U_0} \tag{3-25}$$

Therefore, the X-force is expressed as:

$$X = L \sin\alpha - D\cos\alpha + T \tag{3-26}$$

or

$$X = \frac{1}{2}\rho V^2 S\left(C_L \sin\alpha - C_D \cos\alpha\right) + T \tag{3-27}$$

Neglecting the effect of the w-velocity on the thrust T, by differentiating Equation 3-27 with respect to w and dividing by the mass m, we obtain

$$X_w = \frac{1}{mU_0}\frac{\partial X}{\partial \alpha} = \frac{\rho U_0 S}{2m}\left(C_L \cos\alpha + C_{L_\alpha}\sin\alpha + C_D \sin\alpha - C_{D_\alpha}\cos\alpha\right) \tag{3-28}$$

Now assuming $\alpha$ small we express

$$X_w = \frac{\rho U_0 S}{2m}\left(C_L - C_{D_\alpha}\right) \tag{3-29}$$

where

$$C_{D_\alpha} = \frac{\partial C_D}{\partial \alpha} \tag{3-30}$$

The quantity $C_{D_\alpha}$ represents the change in drag coefficient due to angle of attack changes. When the angle of attack increases, the total drag increases and this

quantity has normally a positive value. It is affected primarily by the wings, and a little by the tail and the fuselage. In aircraft dynamics it is not very important parameter mainly because the changes in angle of attack are small and, hence, its value is very small.

### 5. The $Z_w$ Derivative

It is the rate of change in Z-force due to a w-velocity perturbation divided by the mass m.

$$Z_w = \frac{1}{m}\frac{\partial Z}{\partial w} \tag{3-31}$$

The Z-force similarly is

$$Z = W\cos\theta - L\cos\alpha - D\sin\alpha \tag{3-32}$$

or

$$Z = W\cos\theta - \frac{1}{2}\rho V^2 S\left(C_L\cos\alpha - C_D\sin\alpha\right) \tag{3-33}$$

Neither W or θ depend on w-velocity, since by differentiating Equation 3-33 with respect to w and dividing by the mass m we obtain:

$$Z_w = \frac{1}{mU_0}\cdot\frac{\partial Z}{\partial w} = -\frac{\rho U_0 S}{2m}\left(-C_L\sin\alpha + C_{L_\alpha}\cos\alpha + C_D\cos\alpha + C_{D_\alpha}\sin\alpha\right) \tag{3-34}$$

Now, assuming α small we express

$$Z_w = -\frac{\rho U_0 S}{2m}\left(C_{L_\alpha} + C_D\right) \tag{3-35}$$

where

$$C_{L_\alpha} = \frac{\partial C_L}{\partial \alpha} \tag{3-36}$$

The quantity $C_{L_\alpha}$ represents the change in $C_L$ coefficient due to angle of attack changes. Graphically, it is well-known as "lift curve slope", as shown in Figure 3-3.

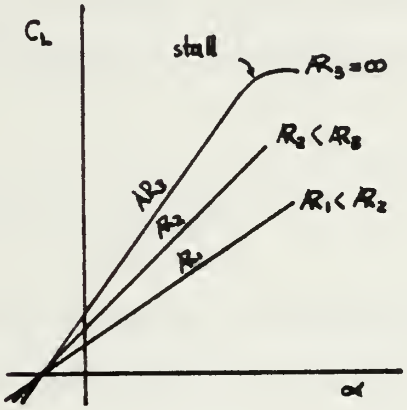


Figure 3-3

Lift Characteristic Curves for Different AR's

The changes in $C_L$, are linear functions of $\alpha$, below the stall. Aspect ratio (AR), is defined simply as the ratio (span/chord) or in more general case as:

$$AR = \frac{(\text{span})^2}{(\text{wing area})} = \frac{b^2}{S} \qquad (3\text{-}37)$$

This ratio, determines the slope of the lift curve. As shown in Figure 3-3, low AR's produce smaller lift coefficient $C_L$ for the same angle of attack $\alpha$.

The $C_{L_\alpha}$ derivative is always positive in the linear region. It is affected primarily by the wings, while the tail and fuselage contribute a little in aircrafts, but largely in missiles. Aeroelastic effects in addition may

cause bending or twisting of the wings that change the angle of attack and hence the value of $C_{L\alpha}$.

### 6. The $M_w$ Derivative

Is the rate of change in the pitching moment M, due to a w-velocity perturbation, divided by the $I_{yy}$ moment of inertia.

$$M_w = \frac{1}{I_{yy}} \cdot \frac{\partial M}{\partial w} \qquad (3\text{-}38)$$

A w-velocity perturbation is associated with an angle of attack $\alpha$ which as we have seen is given by

$$\alpha \cong w/U_0 \qquad (3\text{-}39)$$

When the angle of attack changes from its equilibrium position, the lift of the wing changes in the same sense, and that produces a positive or negative pitching moment depending on the center of gravity location. In addition, the lift of the tail changes and that contributes to the total pitching moment. These contributions together with the fuselage contribution make up this derivative.

We desire a change in angle of attack to produce a pitching moment that tends to restore the original angle of attack. This requires a zero pitching moment in the undisturbed reference flight, a condition known as "trim condition".

The pitching moment, as we have seen, is expressed in terms of the non-dimentional pitching moment coefficient as

$$M = \frac{1}{2}\rho V^2 S c C_M \qquad (3\text{-}40)$$

Because a small change in w-velocity doesn't affect the V velocity, by differentiating Equation 3-40 with respect to w we obtain:

$$\frac{\partial M}{\partial w} = \frac{\partial M}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial w} = \frac{1}{U_0}\cdot\frac{\partial M}{\partial \alpha} = \frac{1}{2U_0}\,\rho V^2 S c\,\frac{\partial C_M}{\partial \alpha} \qquad (3\text{-}41)$$

and finally by letting $V = U_0$ we get

$$M_w = \frac{\rho S U_0 c}{2 I_{yy}} C_{M_\alpha} \qquad (3\text{-}42)$$

The center of gravity location plays a very important role in $C_{M_\alpha}$ derivative. In fact $C_{M_\alpha}$ is proportional to the distance between the center of gravity and the aerodynamic center, the center where the lift force acts.

If the center of gravity coincides with the aerodynamic center, $C_{M_\alpha}$ is zero; if it is ahead of the aerodynamic center, $C_{M_\alpha}$ is negative and the airframe is said to be statically stable; and if it is aft of the aerodynamic center, $C_{M_\alpha}$ is positive and the airframe is said to be statically unstable. The more ahead the center of gravity is located, the steeper the $C_{M_\alpha}$ curve becomes. All the above cases are shown in Figure 3-4.

It is clear that a stable situation like the center of gravity at B or A, an increase in angle of attack from $\alpha_0$ to $\alpha$ produces a negative pitching moment as Figure 3-4 shows, which tends to reduce the angle of attack.

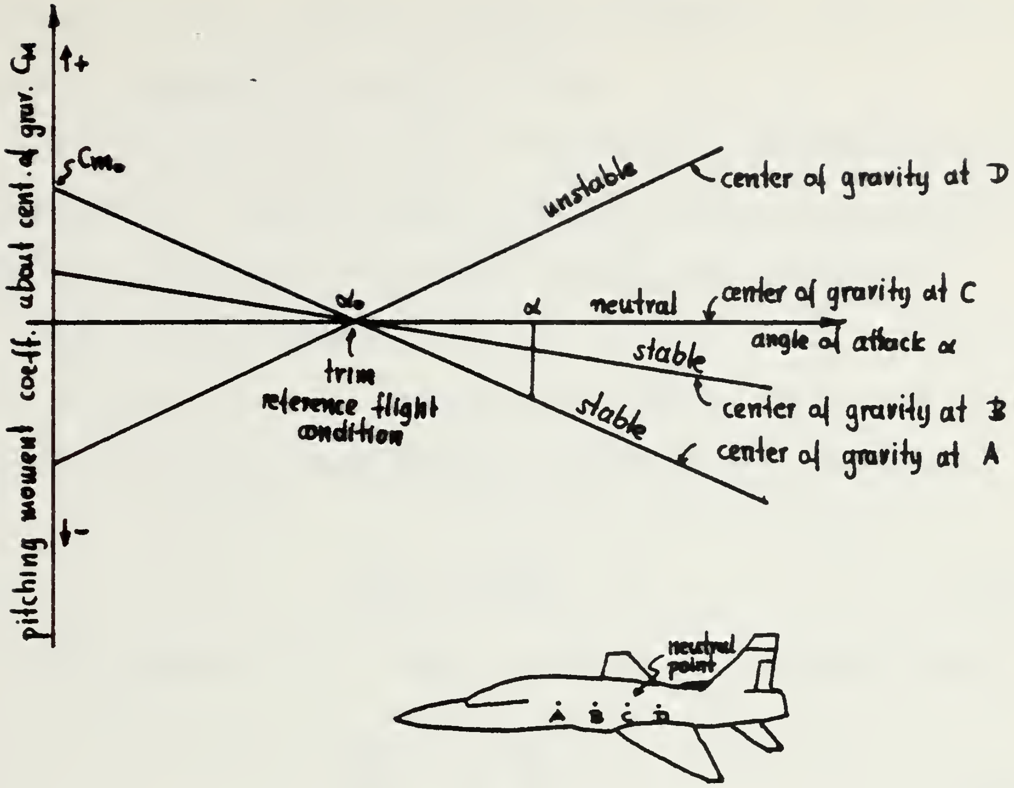


Figure 3-4

$C_{M_\alpha}$ for Different Center of Gravity Locations

Figure 3-4 also shows that for stable situations, for nearly small angles of attack, where the lift is nearly zero a mechanism must be provided to generate a positive $C_{M_0}$. Here comes the effect of the tail which, being a negative angle of attack, generates a small lift force which though multiplied by a larger lever arm produces a balancing positive pitching moment.

In other designs we find the tail being ahead of wings with positive angle of attack. This is a canard

configuration most usually found in missiles, also being the mechanism to generate a positive $C_{M_0}$.

In Figure 3-4 $C_{M_\alpha}$ curves are shown as linear functions of $\alpha$. This is not true for large variations in $\alpha$, but can be nearly true for small variations around $\alpha_0$.

### 7. The $Z_{\dot{w}}$ Derivative

Is the rate of change in Z-force, due to a rate of change in w-velocity perturbation divided by the mass m

$$Z_{\dot{w}} = \frac{1}{m} \cdot \frac{\partial Z}{\partial \dot{w}} \tag{3-43}$$

The relation $\alpha = w/U_0$, leads us to the expression

$$\frac{\partial Z}{\partial \dot{w}} = \frac{1}{U_0} \cdot \frac{\partial Z}{\partial \dot{\alpha}} \tag{3-44}$$

According to Equation 3-16 the Z-force is simplified to

$$Z = W \cos\Theta - L \tag{3-45}$$

Neither $W$ or $\Theta$ depend on rate of change in angle of attack $\dot{\alpha}$, since differentiating Equation 3-45 with respect to $\dot{\alpha}$ we obtain

$$\frac{\partial Z}{\partial \dot{\alpha}} = -\frac{\partial L}{\partial \dot{\alpha}} \tag{3-46}$$

where from Equation 3-1

$$\frac{\partial L}{\partial \dot{\alpha}} = \frac{1}{2} \rho V^2 S \frac{\partial C_L}{\partial \dot{\alpha}} \tag{3-47}$$

Therefore,

$$Z_{\dot{w}} = \frac{1}{m} \cdot \frac{\partial Z}{\partial \dot{w}} = \frac{1}{mU_0} \frac{\partial Z}{\partial \dot{\alpha}} = -\frac{1}{mU_0} \frac{\partial L}{\partial \dot{\alpha}} = \frac{-1}{mU_0} \frac{1}{2} \rho V^2 S \frac{\partial C_L}{\partial \dot{\alpha}} \tag{3-48}$$

To form a non-dimensional coefficient, we multiply and divide Equation 3-48 by c/2Uo so that letting V = Uo we get

$$Z_{\dot{w}} = -\frac{1}{mU_0} \cdot \frac{1}{2} \rho U_0^2 S \frac{c}{2U_0} \cdot \frac{\partial C_L}{\partial(\dot{\alpha} c/2U_0)} \qquad (3\text{-}49)$$

or

$$Z_{\dot{w}} = -\frac{\rho S c}{4m} C_{L_{\dot{\alpha}}} \qquad (3\text{-}50)$$

where

$$C_{L_{\dot{\alpha}}} = \frac{\partial C_L}{\partial(\dot{\alpha} c/2U_0)} \qquad (3\text{-}51)$$

The quantity $C_{L_{\dot{\alpha}}}$ represents a change in lift coefficient due to a variation in the rate of change of the angle of attack $\dot{\alpha}$.

If the angle of attack changes rapidly, the pressure distribution on the wing, tail and fuselage, does not adjust itself instantaneously to its equilibrium value. This, a change in lift and drag occurs to the sudden changes in angle of attack.

Its sign is positive for low speeds but can be positive or negative on high speeds depending on aeroelastic effects.

Also, a sudden change in angle of attack alters the downwash field as seen by the tail and a time delay is elapsed before the tail senses this alteration.

All those effects are combined to give rise in the $C_{L_{\dot{\alpha}}}$ derivative.

### 8. The $M_{\dot{w}}$ Derivative

Is the rate of change in the pitching moment M, due to a rate of change in w-velocity perturbation divided by the $I_{yy}$ moment of inertia.

$$M_{\dot{w}} = \frac{1}{I_{yy}} \cdot \frac{\partial M}{\partial \dot{w}} \tag{3-52}$$

This derivative $\partial M/\partial \dot{w}$ can be expressed as:

$$\frac{\partial M}{\partial \dot{w}} = \frac{1}{U_o} \cdot \frac{\partial M}{\partial \dot{\alpha}} \tag{3-53}$$

According to Equation 3-3

$$M = \frac{1}{2} \rho V^2 S c C_M \tag{3-54}$$

Since V is independent of the rate of change in angle of attack $\dot{\alpha}$, by differentiating Equation 3-54 with respect to $\dot{\alpha}$ we obtain:

$$\frac{\partial M}{\partial \dot{\alpha}} = \frac{1}{2} \rho V^2 S c \frac{\partial C_M}{\partial \dot{\alpha}} \tag{3-55}$$

Therefore,

$$M_{\dot{w}} = \frac{1}{I_{yy}} \cdot \frac{\partial M}{\partial \dot{w}} = \frac{1}{U_o I_{yy}} \cdot \frac{\partial M}{\partial \dot{\alpha}} = \frac{1}{U_o I_{yy}} \cdot \frac{1}{2} \rho V^2 S c \frac{\partial C_M}{\partial \dot{\alpha}} \tag{3-56}$$

To form a non-dimensional coefficient, we multiply and divide Equation 3-56 by c/2Uo so that letting V = Uo, we get

$$M_{\dot{w}} = \frac{1}{U_o I_{yy}} \cdot \frac{1}{2} \rho U_o^2 S c \frac{c}{2U_o} \cdot \frac{\partial C_M}{\partial (\dot{\alpha} c/2U_o)} \tag{3-57}$$

or

$$M_{\dot{w}} = \frac{\rho S c^2}{4 I_{yy}} C_{M\dot{\alpha}} \tag{3-58}$$

where

$$C_{M_{\dot{\alpha}}} = \frac{\partial C_M}{\partial(\dot{\alpha}c/2U_0)} \qquad (3\text{-}59)$$

For quantity $C_{M_{\dot{\alpha}}}$ represents a change in pitching moment coefficient due to a variation in the rate of change of the angle of attack $\dot{\alpha}$.

The change in lift that is produced by sudden changes in angle of attack discussed already, multiplied by corresponding lever arms, generates changes in pitching moment and that is why $C_{M_{\dot{\alpha}}}$ arises.

Same lag effects in the downwash field discussed, contribute to this derivative, as well as aeroelastic effects, that also disturb the air stream of the wing and tail, so that the pressure distribution over the aerodynamic surfaces does not adjust instantaneously.

The sign of this derivative is negative and is relatively important in longitudinal dynamics. Large absolute values of this derivative are desired because it causes damping.

9. The Zq Derivative

Is the rate of change in Z-force, due to a q-pitching velocity perturbation divided by the mass m

$$Z_q = \frac{1}{m}\frac{\partial Z}{\partial q} \qquad (3\text{-}60)$$

According to Equation 3-16, the Z-force is simplified to:

$$Z = W\cos\Theta - L \qquad (3\text{-}61)$$

Neither W or $\theta$ depend on the pitching velocity (q), since differentiating Equation 3-61 with respect to q, we obtain:

$$\frac{\partial Z}{\partial q} = -\frac{\partial L}{\partial q} \qquad (3\text{-}62)$$

where from Equation 3-1

$$\frac{\partial L}{\partial q} = \frac{1}{2}\rho V^2 S \frac{\partial C_L}{\partial q} \qquad (3\text{-}63)$$

Therefore:

$$Z_q = \frac{1}{m}\frac{\partial Z}{\partial q} = -\frac{1}{m}\cdot\frac{\partial L}{\partial q} = -\frac{1}{m}\,\frac{1}{2}\rho V^2 S \frac{\partial C_L}{\partial q} \qquad (3\text{-}64)$$

To form a non-dimensional coefficient we multiply and divide Equation 3-64 by c/2Uo so that letting V=Uo we get:

$$Z_q = -\frac{1}{m}\cdot\frac{1}{2}\rho U_o^2 S \frac{c}{2U_o}\cdot\frac{\partial C_L}{\partial (qc/2U_o}} \qquad (3\text{-}65)$$

or

$$Z_q = -\frac{\rho S U_o c}{4m} C_{L_q} \qquad (3\text{-}66)$$

where

$$C_{L_q} = \frac{\partial C_L}{\partial (qc/2U_o)} \qquad (3\text{-}67)$$

The quantity $C_{L_q}$ represents a change in lift coefficient due to a variation in the pitching velocity.

If the airframe pitches about its center of gravity, the angle of attack changes and a change in left of wing, tail and fuselage is developed that gives rise to this derivative.

If has a small positive value in low speed flight but may have large values positive or negative in high speeds due to aeroelastic effects.

Curved longitudinal flight paths give rise also to centrifugal forces on all components of the airframe that due to aeroelastic effects cause a bending of aerodynamic surfaces and, hence, a change in lift.

In the past without computer solutions available, this derivative was considered zero to simplify the analysis.

### 10. The Mq Derivative

Is the rate of change in the pitching moment M, due to a pitching velocity perturbation, divided by the Iyy moment of inertia.

$$M_q = \frac{1}{I_{yy}} \cdot \frac{\partial M}{\partial q} \tag{3-68}$$

According to Equation 3-3,

$$M = \frac{1}{2} \rho V^2 S c C_M \tag{3-69}$$

Since V is independent of the pitching velocity q, by differentiating Equation 3-69 with respect to q, we obtain:

$$\frac{\partial M}{\partial q} = \frac{1}{2} \rho V^2 S c \frac{\partial C_M}{\partial q} \tag{3-70}$$

Therefore,

$$M_q = \frac{1}{I_{yy}} \cdot \frac{\partial M}{\partial q} = \frac{1}{2 I_{yy}} \cdot \rho V^2 S c \frac{\partial C_M}{\partial q} \tag{3-71}$$

To form a non-dimensional coefficient, we multiply and divide Equation 3-71 by c/2Uo, so that letting V=Uo we get:

$$M_q = \frac{1}{2I_{yy}} \rho U_o^2 S c \, \frac{c}{2U_o} \cdot \frac{\partial C_M}{\partial(qc/2U_o)} \qquad (3\text{-}72)$$

or

$$M_q = \frac{\rho U_o S c^2}{4 I_{yy}} C_{M_q} \qquad (3\text{-}73)$$

where

$$C_{M_q} = \frac{\partial C_M}{\partial(qc/2U_o)} \qquad (3\text{-}74)$$

The quantity $C_{M_q}$ represents a change in pitching moment coefficient due to a variation in the pitching velocity q.

The change in lift that was produced, due to pitching velocity perturbations discussed already, multiplied by corresponding lever arms, generates changes in pitching moment and that is why $C_{M_q}$ rises.

The tail is the main contributor to this derivative which is also referred to as "pitching damping derivative". Aeroelastic effects and centrigugal forces generated due to curved longitudinal flight paths also contribute to this derivative.

In longitudinal dynamics $C_{M_q}$ is very important. It has a negative value, and the larger it is, the more the

damping is increased; a condition that is highly desired. In high speed flights $C_{M_q}$ may have positive or negative value depending on aeroelastic effects. It is also noted that tailless airplanes have a poor damping.

### 3.3 LATERAL - DIRECTIONAL STABILITY DERIVATIVES

Lateral-Directional stability derivatives are encountered in the lateral directional equations and, hence, they uniquely specify the Y-force, the L-rolling moment and the N-yawing moment.

Dimensional analysis approach indicates that we can express the lateral-directional aerodynamic forces and moments in terms of their corresponding non-dimensional coefficients as:

Y-force $Y = C_y \frac{1}{2} \rho V^2 S$ (3-75)

L-rolling moment $L = C_\ell \frac{1}{2} \rho V^2 S b$ (3-76)

N-yawing moment $N = C_n \frac{1}{2} \rho V^2 S b$ (3-77)

where $C_y$ , $C_\ell$ , $C_n$ are the corresponding Y-force, rolling moment and yawing moment coefficients, and b is a reference length, usually taken the wing span.

The Lateral-Directional Stability Derivatives can be evaluated as follows:

1. The Yv Derivative

Is the rate of change in Y-force due to a v-velocity perturbation divided by the mass m:

$$Y_v = \frac{1}{m} \cdot \frac{\partial Y}{\partial v} \qquad (3\text{-}78)$$

A v-velocity perturbation causes a small change in the angle of sideslip $\beta$ which is expressed according to Equation 2-133, approximately as:

$$\beta \cong v/U_o \qquad (3\text{-}79)$$

so that

$$Y_v = \frac{1}{m U_o} \cdot \frac{\partial Y}{\partial \beta} \qquad (3\text{-}80)$$

Assuming that the V-velocity is independent of the angle of sideslip $\beta$, differentiating Equation 3-75 with respect to $\beta$, we obtain:

$$\frac{\partial Y}{\partial \beta} = \frac{1}{2} \rho V^2 S \frac{\partial C_y}{\partial \beta} \qquad (3\text{-}81)$$

Therefore, letting V=Uo we get:

$$Y_v = \frac{\rho U_o S}{2m} C_{y_\beta} \qquad (3\text{-}82)$$

where

$$C_{y_\beta} = \partial C_y / \partial \beta \qquad (3\text{-}83)$$

The quantity $C_{y_\beta}$ represents the change in the $C_y$ side force coefficient due to a change in the angle of sideslip $\beta$.

The main contributor of this derivative is the vertical tail. Secondarily the fuselage, and also the wings contribute a little.

Figure 3-5 shows the forces that the airframe experiences due to a v-velocity perturbation. First an angle of sideslip builds up and all aerodynamic surfaces are sensing a change in the approaching free stream.

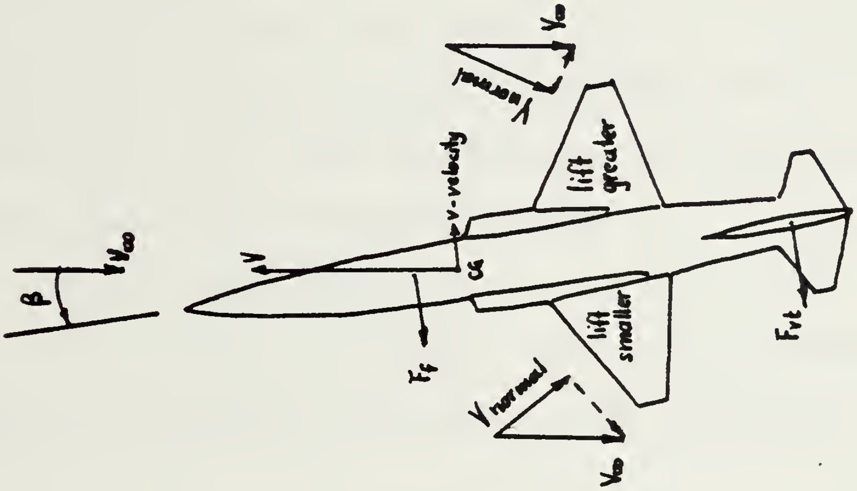


Figure 3-5

Forces Experienced Due to a v-Velocity Perturbation

As a consequence, a force $F_f$ due to the fuselage and a force $F_{vt}$ due to the vertical tail appears. The wings also experience a different lift force. For the configuration shown in Figure 3-5, the right wing experiences more lift than the left wing because it senses a greater V-normal component of the approaching free stream. Those lift forces may contribute to a small change in the Y-force.

All these contributions make a rise to $C_{y_\beta}$ derivative which is very important in lateral-directional dynamics. It

introduces a damping that tends to eliminate the angle $\beta$, because the resultant Y-force appears acting on a point aft of CG location and that generates the appropriate moment to do this.

Its value is negative and it might be reasonable to desire a large value, for the $\beta$ angle to die out quickly. Although high maneuverability airplanes must have a small negative value for this derivative, if they were to perform easy roll maneuvers and banked turns.

### 2. The $L_v$ Derivative

Is the rate of change in L-rolling moment due to a v-velocity perturbation divided by the Ixx moment of inertia.

$$L_v = \frac{1}{I_{xx}} \cdot \frac{\partial L}{\partial v} \tag{3-84}$$

It is also clearly expressed as:

$$L_v = \frac{1}{U_o I_{xx}} \cdot \frac{\partial L}{\partial \beta} \tag{3-85}$$

Differentiating Equation 3-76, with respect to $\beta$, we obtain

$$\frac{\partial L}{\partial \beta} = \frac{1}{2} \rho V^2 S b \frac{\partial C_l}{\partial \beta} \tag{3-86}$$

Therefore, letting V=Uo we get:

$$L_v = \frac{\rho U_o S b}{2 I_{xx}} C_{l_\beta} \tag{3-87}$$

where

$$C_{\ell_\beta} = \frac{\partial C_\ell}{\partial \beta} \tag{3-88}$$

The quantity $C_{\ell_\beta}$ represents the change in the $C_\ell$ rolling moment coefficient due to a change in the angle of sideslip $\beta$.

It is usually referred to as the effective Dihedral Derivative.

Dihedral is often used as a mechanism to improve lateral-directional stability. Figure 3-6(a) shows a head-on view of an airplane whose wings are tilted up at some dihedral angle with respect to the horizontal.

We have already discussed that due to a v-velocity perturbation, according to the configuration shown in Figure 3-5, where a positive sideslip angle $\beta$ is shown, right wing generates more lift than the left. This is true for the swept wings. If the wings are not swept, the mechanism to produce the same effect is the dihedral. Usually even swept wings are given a small dihedral to improve lateral-directional stability.

To understand the dihedral effect, consider the aircraft shown in Figure 3-7 with non-swept wings but with some dihedral angle. The aircraft is distributed by an angle of sideslip $\beta$.

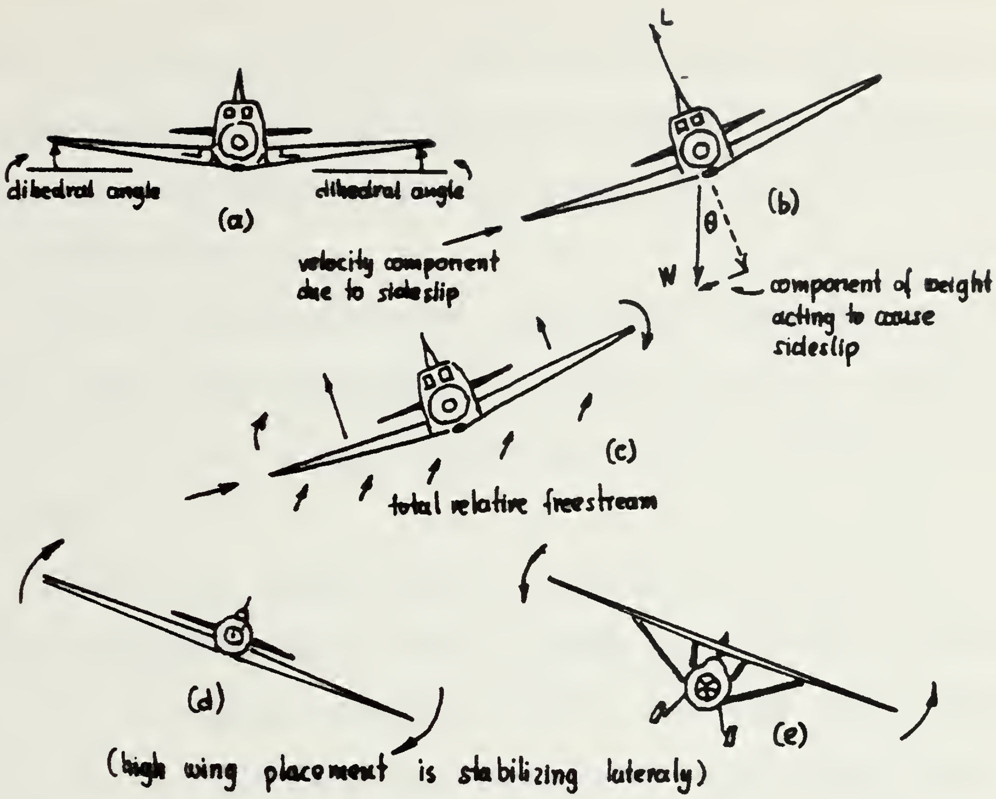

L
dihedral angle
dihedral angle
(a)
(b)
θ
W
velocity component due to sideslip
component of weight acting to cause sideslip
(c)
total relative freestream
(d)
(e)
(high wing placement is stabilizing lateraly)

Figure 3-6

Dihedral Effect on Lateral-Directional Stability

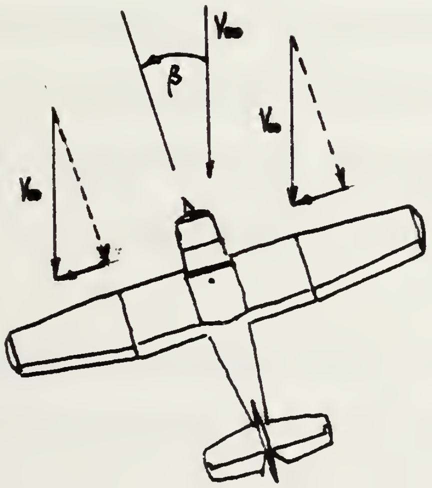


Figure 3-7

Dihedral Effect on Non-Swept Wings

The perpendicular components of the free stream velocity are equal but the angle of attack that the right wing projects to this perpendicular component is greater than the angle of attack that the left wing projects, as an effect of the dihedral angle.

As a consequence, the right wing generates more lift than the left wing as shown in Figure 3-6(c). Those lift forces contribute to a change in the rolling moment coefficient.

Figure 3-6(d) and (c) show the effect of the high-wing placement which has the same effect as the dihedral angle. It is clearly shown that in the low wing placement configuration, the fuselage does not permit the vertical component of the free stream to act upon the right wing to produce a stabilizing moment, whereas in the high wing placement configuration the fuselage effect is nearly zero and that makes a stabilizing rolling moment to show up.

In addition, the force acting on the vertical tail shown in Figure 3-5, multiplied by the corresponding vertical arm relative to the x-longitudinal axis, contributes also to the same effect.

$C_{\ell_\beta}$ is nearly always negative in sign and the corresponding rolling moment L, is negative for a positive sideslip angle $\beta$. It is a very important derivative in lateral directional dynamics because it causes a damping and

improves the stability. Although high negative values of this derivative will tend to decrease the maneuverability of the airplane in roll movements. In missiles a low negative value of this derivative is desired.

### 3. The Nv Derivative

Is the rate of change in N-yawing moment due to a v-velocity perturbation divided by the Izz moment of inertia.

$$N_v = \frac{1}{I_{zz}} \cdot \frac{\partial N}{\partial v} \qquad (3\text{-}89)$$

It is also clearly expressed as:

$$N_v = \frac{1}{U_o I_{zz}} \cdot \frac{\partial N}{\partial \beta} \qquad (3\text{-}90)$$

Differentiating Equation 3-77, with respect to $\beta$, we obtain:

$$\frac{\partial N}{\partial \beta} = \frac{1}{2} \rho V^2 S b \frac{\partial C_n}{\partial \beta} \qquad (3\text{-}91)$$

Therefore, letting V=Uo we get:

$$N_v = \frac{\rho U_o S b}{2 I_{zz}} C_{n_\beta} \qquad (3\text{-}92)$$

where

$$C_{n_\beta} = \partial C_n / \partial \beta \qquad (3\text{-}93)$$

The quantity $C_{n_\beta}$ represents the change in the $C_n$ yawing moment coefficient due to a change in the angle of sideslip $\beta$.

It is usually referred to as the Weathercock Derivative. The main contributor of this derivative is the vertical tail and secondarily, the fuselage. If the airframe experiences a v-velocity perturbation and an angle of sideslip $\beta$ appears, the free relative stream strikes the fuselage and the vertical tail in such a manner that the force $F_f$ , shown in Figure 3-5, produces a destabilizing yawing moment and the force $F_{vt}$ a stabilizing moment. Wings contribute a little in the positive (stabilizing) sense. The total value of this derivative is positive, signifying static directional stability.

The $C_{n_\beta}$ derivative, is very important in lateral-directional dynamics. The vertical tail must be properly designed to provide a sufficient weathercock effect. At large, sideslip angles a stall may occur due to flow separation and a catastrophic sideslip divergence may result. A low aspect ratio prevents stalling but decreases stability.

Usually a large positive value of the $C_{n_\beta}$ derivative is desired, by extending the vertical tail. A large value helps the pilot in good coordinated turns and prevents sideslip.

4. The $Y_{\dot{v}}$ Derivative

Is the rate of change in Y-force, due to a rate of change in v-velocity perturbation, divided by the mass m.

$$Y_{\dot{v}} = \frac{1}{m} \cdot \frac{\partial Y}{\partial \dot{v}} \qquad (3\text{-}94)$$

It is also clearly expressed as:

$$Y_{\dot{v}} = \frac{1}{U_o m} \cdot \frac{\partial Y}{\partial \dot{\beta}} \tag{3-95}$$

Differentiating Equation 3-75 with respect to $\dot{\beta}$ we obtain:

$$\frac{\partial Y}{\partial \dot{\beta}} = \frac{1}{2} \rho V^2 S \frac{\partial C_y}{\partial \dot{\beta}} \tag{3-96}$$

To form a non-dimensional coefficient we multiply and divide Equation 3-96 by b/2Uo.

$$\frac{\partial Y}{\partial \dot{\beta}} = \frac{1}{2} \rho V^2 S \frac{b}{2U_o} \cdot \frac{\partial C_y}{\partial (\dot{\beta} b/2U_o)} \tag{3-97}$$

Therefore, letting V=Uo we get:

$$Y_{\dot{v}} = \frac{\rho S b}{4m} C_{y_{\dot{\beta}}} \tag{3-98}$$

where

$$C_{y_{\dot{\beta}}} = \frac{\partial C_y}{\partial (\dot{\beta} b/2U_o)} \tag{3-99}$$

The quantity $C_{y_{\dot{\beta}}}$ represents the change in the coefficient with variation in rate of change of the sideslip angle $\beta$.

If the sideslip angle $\beta$ changes rapidly, the pressure distribution on the vertical tail and fuselage does not adjust itself instantaneously to its equilibrium value.

This effect, as well as aeroelastic effects, give rise in $C_{y_{\dot{\beta}}}$ derivative but its value is very small and is usually neglected.

### 5. The $L_{\dot{v}}$ Derivative

Is the rate of change in L-rolling moment, due to a rate of change in v-velocity perturbation divided by the Ixx moment of inertia.

$$L_{\dot{v}} = \frac{1}{I_{xx}} \cdot \frac{\partial L}{\partial \dot{v}} \qquad (3\text{-}100)$$

It is also clearly expressed as:

$$L_{\dot{v}} = \frac{1}{U_o I_{xx}} \cdot \frac{\partial L}{\partial \dot{\beta}} \qquad (3\text{-}101)$$

Differentiating Equation 3-76, with respect to $\dot{\beta}$, we obtain:

$$\frac{\partial L}{\partial \dot{\beta}} = \frac{1}{2} \rho V^2 S b \frac{\partial C_\ell}{\partial \dot{\beta}} \qquad (3\text{-}102)$$

To form a non-dimensional coefficient

$$\frac{\partial L}{\partial \dot{\beta}} = \frac{1}{2} \rho V^2 S b \, \frac{b}{2U_o} \cdot \frac{\partial C_L}{\partial(\dot{\beta} b/2U_o)} \qquad (3\text{-}103)$$

Therefore, letting V=Uo

$$L_{\dot{v}} = \frac{\rho S b^2}{4 I_{xx}} C_{\ell_{\dot{\beta}}} \qquad (3\text{-}104)$$

where

$$C_{\ell_{\dot{\beta}}} = \frac{\partial C_\ell}{\partial(\dot{\beta} b/2U_o)} \qquad (3\text{-}105)$$

The quantity $C_{\ell_{\dot{\beta}}}$ represents the change in the $C_\ell$ rolling moment coefficient with variation in rate of change of the sideslip angle $\beta$.

The change in the side forces on the vertical tail and the fuselage produced due to sudden changes in the sideslip angle $\beta$, discussed already, multiplied by corresponding lever arms provide a total change in the $C_\ell$ rolling moment coefficient.

This derivative has a very small value because the lever arms are very small and it is usually neglected.

### 6. The $N_{\dot v}$ Derivative

Is the rate of change in the N-yawing moment, due to a rate of change in v-velocity perturbation divided by the Izz moment of inertia.

$$N_{\dot v} = \frac{1}{I_{zz}} \cdot \frac{\partial N}{\partial \dot v} \qquad (3\text{-}106)$$

It is also clearly expressed as:

$$N_{\dot v} = \frac{1}{U_o I_{zz}} \cdot \frac{\partial N}{\partial \dot\beta} \qquad (3\text{-}107)$$

Differentiating Equation 3-77, with respect to $\dot\beta$, we obtain:

$$\frac{\partial N}{\partial \dot\beta} = \frac{1}{2} \rho V^2 S b \frac{\partial C_n}{\partial \dot\beta} \qquad (3\text{-}108)$$

To form a non-dimensional coefficient:

$$\frac{\partial N}{\partial \dot\beta} = \frac{1}{2} \rho V^2 S b \frac{b}{2U_o} \cdot \frac{\partial C_n}{\partial (\dot\beta b/2U_o)} \qquad (3\text{-}109)$$

Therefore, letting V=Uo we get:

$$N_{\dot v} = \frac{\rho S b^2}{4 I_{zz}} C_{n_{\dot\beta}} \qquad (3\text{-}110)$$

where

$$C_{n_{\dot{\beta}}} = \frac{\partial C_n}{\partial(\dot{\beta} b / 2V_o)} \tag{3-111}$$

The quantity $C_{n_{\dot{\beta}}}$ represents the change in the $C_n$ yawing moment coefficient with variation in rate of change of the sideslip angle $\beta$.

The change in the side forces on the vertical tail and the fuselage produced due to sudden changes in the sideslip angle $\beta$, discussed already, multiplied by corresponding lever arms provide a total change in the $C_n$ yawing moment coefficient.

The lever arm of the vertical tail this time is long and hence, this derivative may be considerable. Positive values of this derivative increase the damping in the yawing motion and are desirable in general.

7. The $Y_p$ Derivative

Is the rate of change in the Y-force, due to a p-rolling velocity perturbation divided by the mass m.

$$Y_p = \frac{1}{m} \cdot \frac{\partial Y}{\partial p} \tag{3-112}$$

Differentiating Equation 3-75, with respect to p, we obtain:

$$\frac{\partial Y}{\partial p} = \frac{1}{2} \rho V^2 S \frac{\partial C_y}{\partial p} \tag{3-113}$$

To form a non-dimensional coefficient

$$\frac{\partial Y}{\partial p} = \frac{1}{2} \rho V^2 S \frac{b}{2U_0} \cdot \frac{\partial C_y}{\partial (pb/2U_0)} \tag{3-114}$$

Therefore, letting V=Uo

$$Y_p = \frac{\rho U_0 S b}{4m} C_{y_p} \tag{3-115}$$

where

$$C_{y_p} = \frac{\partial C_y}{\partial (pb/2U_0)} \tag{3-116}$$

The quantity $C_{y_p}$ represents the change in the $C_y$ Y-force coefficient with variation in the rolling velocity p.

When an aircraft has a roll velocity p, a force is acting on the vertical tail that usually opposes the roll movement. This consists the main contribution to this derivative and is small enough to be neglected.

The force acting on the vertical fin may be affected by the sidewash from the wing that gives a slightly different direction of the airstream.

### 8. The $L_p$ Derivative

Is the rate of change in the L-rolling moment, due to a rolling velocity p perturbation divided by the Ixx moment of inertia.

$$L_p = \frac{1}{I_{xx}} \cdot \frac{\partial L}{\partial p} \tag{3-117}$$

Differentiating Equation 3-76, with respect to p, we obtain:

$$\frac{\partial L}{\partial p} = \frac{1}{2}\rho V^2 S b \frac{\partial C_\ell}{\partial p} \qquad (3\text{-}118)$$

To form a non-dimensional coefficient,

$$\frac{\partial L}{\partial p} = \frac{1}{2}\rho V^2 S b \frac{b}{2U_0} \cdot \frac{\partial C_\ell}{\partial (pb/2U_0)} \qquad (3\text{-}119)$$

Therefore, letting V=Uo

$$L_p = \frac{\rho U_0 S b^2}{4 I_{xx}} C_{\ell_p} \qquad (3\text{-}120)$$

where

$$C_{\ell_p} = \frac{\partial C_\ell}{\partial (pb/2U_0)} \qquad (3\text{-}121)$$

The quantity $C_{\ell_p}$ represents the change in $C_\ell$ rolling moment coefficient with variation in the rolling velocity p.

This derivative is also referred to as "roll damping derivative", because it has usually a negative or damping effect coming mainly from the wings and secondarily, from the vertical and horizontal tails.

When the aircraft rolls, the force acting on the vertical tail, shown in Figure 3-8, multiplied by the corresponding lever arm, produces an opposing rolling moment. In addition, on the down-going wing, the effective angle of attack is increased and on the up-going wing, decreased. This makes the lift on the down-going wing greater and thus an opposing rolling moment is produced. Similar effect, on a smaller scale, is produced by the horizontal tails.

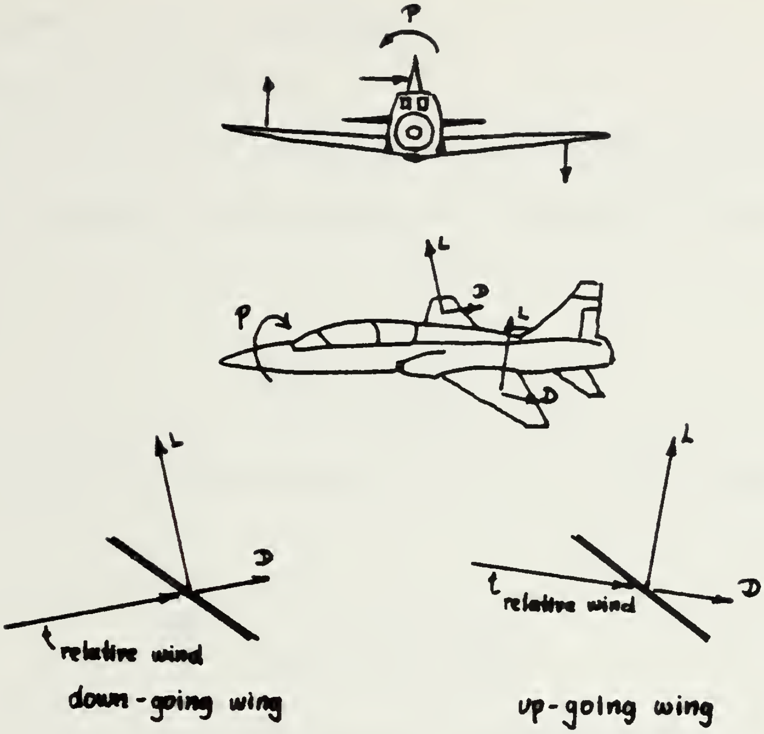


Figure 3-8

Forces Due to Rolling Velocity (p)

This is true as long as the down-going wing is not flying near the stall due to a large angle of attack experienced in some cases. In such cases a positive effect will result in autorotation.

The derivative $C_{\ell_p}$ is quite important in lateral-directional dynamics. The value of this derivative directly affects the ailerons design. A small negative value of this derivative will result in a better aileron input

response while a large negative value will make the airframe heavily damped in roll maneuvers.

9. The $N_p$ Derivative

Is the rate of change in the N-yawing moment, due to a rolling velocity p perturbation, divided by the Izz moment of inertia.

$$N_p = \frac{1}{I_{zz}} \cdot \frac{\partial N}{\partial p} \tag{3-122}$$

Differentiating Equation 3-77, with respect to p, we obtain:

$$\frac{\partial N}{\partial p} = \frac{1}{2} \rho V^2 S b \frac{\partial C_n}{\partial p} \tag{3-123}$$

To form a non-dimensional coefficient,

$$\frac{\partial N}{\partial p} = \frac{1}{2} \rho V^2 S b \frac{b}{2U_0} \cdot \frac{\partial C_n}{\partial (pb/2U_0)} \tag{3-124}$$

Therefore, letting V=Uo

$$N_p = \frac{\rho U_0 S b^2}{4 I_{zz}} C_{n_p} \tag{3-125}$$

where

$$C_{n_p} = \frac{\partial C_n}{\partial (pb/2U_0)} \tag{3-126}$$

The quantity $C_{n_p}$ represents the change in $C_n$ yawing moment coefficient with variation in the folling velocity p.

This derivative usually has a negative or positive effect coming from the wings, the vertical tail and secondarily, from the horizontal tail.

When the aircraft rolls, the force acting on the vertical tail, shown in Figure 3-8, multiplied by the corresponding lever arm produces a yawing moment towards the same side of the roll motion, i.e., roll to starboard causes yaw to starboard. This contributes a positive effect and depands upon the vertical tail geometry and the sidewash from the wings.

In addition, the contribution of the wings can be considered to be made of two parts, one opposite the other, the total effect being negative. When the airframe rolls, on the down-going wing, the effective angle of attack increases and on the up-going wing decreases, as shown in Figure 3-8. As a consequence, the lift and drag are greater on the down-going wing than on the up-going wing. The projections of the drag forces on the $xoy$ plane usually produce a yawing moment towards the same side of the roll motion. This produces a positive effect while the projections of the lift forces produce a yawing moment towards the opposite side of the roll motion, and this produces a negative or damping effect.

The overall value of this derivative is usually negative, i.e. roll to starboard produces a yaw to port. At higher speeds the numerical value decreases and may become positive.

This derivative is fairly important in lateral-directional dynamics, as far as roll maneuvers and their effect to yaw.

### 10. The $Y_r$ Derivative

Is the rate of change in the Y-side force, due to a yawing velocity perturbation r, divided by the mass m.

$$Y_r = \frac{1}{m} \cdot \frac{\partial Y}{\partial r} \tag{3-127}$$

Differentiating Equation 3-75, with respect to r, we obtain:

$$\frac{\partial Y}{\partial r} = \frac{1}{2} \rho V^2 S \frac{\partial C_y}{\partial r} \tag{3-128}$$

To form a non-dimensional coefficient,

$$\frac{\partial Y}{\partial r} = \frac{1}{2} \rho V^2 S \frac{b}{2U_0} \cdot \frac{\partial C_y}{\partial (rb/2U_0)} \tag{3-129}$$

Therefore, letting V=Uo

$$Y_r = \frac{\rho U_0 S b}{4m} C_{y_r} \tag{3-130}$$

where

$$C_{y_r} = \frac{\partial C_y}{\partial (rb/2U_0)} \tag{3-131}$$

The quantity $C_{y_r}$ represents the change in $C_y$ Y-force coefficient with variation in the yawing velocity r.

When the aircraft has a yaw velocity r, as Figure 3-9 shows, a force is acting on the fuselage that usually has a

negative or damping effect on the yaw motion and a force is acting on the vertical tail that usually has the same effect.

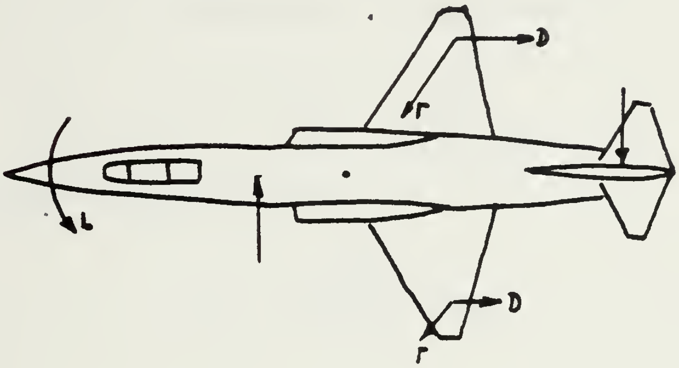


Figure 3-9

Forces Due to Yaw Velocity (r)

Both contributions are small enough to be neglected, therefore, this derivative is of little importance in lateral-directional dynamics.

11. The $L_r$ Derivative

Is the rate of change in the L-rolling moment, due to a yawing velocity r perturbation, divided by the Ixx moment of inertia.

$$L_r = \frac{1}{I_{xx}} \cdot \frac{\partial L}{\partial r} \qquad (3\text{-}132)$$

Differentiating Equation 3-76, with respect to r, we obtain:

$$\frac{\partial L}{\partial r} = \frac{1}{2} \rho V^2 S b \frac{\partial C_\ell}{\partial r} \tag{3-133}$$

To form a non-dimensional coefficient,

$$\frac{\partial L}{\partial r} = \frac{1}{2} \rho V^2 S b \frac{b}{2U_o} \cdot \frac{\partial C_\ell}{\partial (rb/2U_o)} \tag{3-134}$$

Therefore, letting V=Uo

$$L_r = \frac{\rho U_o S b^2}{4 I_{xx}} C_{\ell_r} \tag{3-135}$$

where

$$C_{\ell_r} = \frac{\partial C_\ell}{\partial (rb/2U_o)} \tag{3-136}$$

The quantity $C_{\ell_r}$ represents the change in $C_\ell$ rolling moment coefficient with a variation in the yawing velocity r.

When the aircraft is yawing about its vertical axis, the outer wing moves faster than the inner. As a consequence, the lift on the outer wing is greater than that of the inner and, hence, a rolling moment is produced towards the same side of the yaw motion. This contributes a positive effect, i.e. yaw to starboard produces roll to starboard.

In addition to this major wing contribution, the force acting on the vertical tail multiplied by the corresponding lever arm, produces the same rolling effect.

The total value of this derivative is usually positive, but it is not so important in lateral-directional dynamics.

12. The $N_r$ Derivative

Is the rate of change in the N-yawing moment, due to a yawing velocity r perturbation, divided by the Izz moment of inertia.

$$N_r = \frac{1}{I_{zz}} \cdot \frac{\partial N}{\partial r} \qquad (3\text{-}137)$$

Differentiating Equation 3-77, with respect to r, we obtain:

$$\frac{\partial N}{\partial r} = \frac{1}{2} \rho V^2 S b \frac{\partial C_n}{\partial r} \qquad (3\text{-}138)$$

To form a non-dimensional coefficient,

$$\frac{\partial N}{\partial r} = \frac{1}{2} \rho V^2 S b \frac{b}{2U_0} \cdot \frac{\partial C_n}{\partial (rb/2U_0)} \qquad (3\text{-}139)$$

Therefore, letting V=Uo

$$N_r = \frac{\rho U_0 S b^2}{4 I_{zz}} C_{n_r} \qquad (3\text{-}140)$$

where

$$C_{n_r} = \frac{\partial C_n}{\partial (rb/2U_0)} \qquad (3\text{-}141)$$

The quantity $C_{n_r}$ represents the change in $C_n$ yawing moment coefficient, with a variation in the yawing velocity r.

This derivative is known as the "yaw damping derivative", because it usually has a negative effect.

The main contributor is the vertical tail. When the aircraft is yawing, the force acting on the vertical tail multiplied by the corresponding long lever arm, produces a considerable yawing moment that opposes the initial yawing disturbance.

In addition, the drag on the outer wing is greater than that of the inner wing. That produces a yawing moment that opposes the initial yawing disturbance. The force on the fuselage also contributes an opposing effect.

This derivative is very important in lateral-directional dynamics. Its total effect is negative and large values are desired for more effective damping.

### 3.4 CONTROL SURFACE DEFLECTIONS

We have discussed so far all the important stability derivatives that rise due to the airframe geometry. They uniquely specify the forces and moments about the center of gravity provided that they are multiplied by the correct factor that depends upon the airstream and airframe characteristics.

A stability analysis is possible now for determining the longitudinal and lateral directional modes of motion, corresponding to an initial disturbance, i.e. homogeneous solution to the differential equations.

In addition, some means of control in longitudinal and lateral directional motion are available to the human or automatic pilot. They are shown in Figure 3-10.

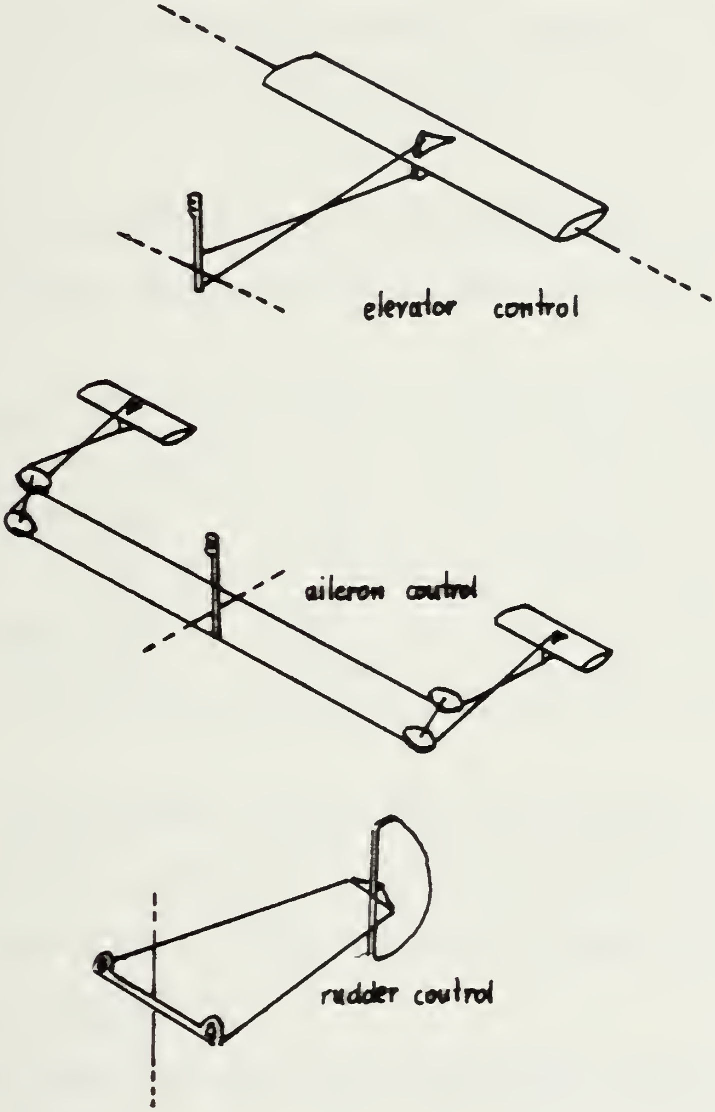


Figure 3-10

Basic Longitudinal and Lateral Control Systems

Elevator, controls the longitudinal motion of the airframe providing nose up or nose down moments as a response to corresponding control surface deflections.

Aileron and rudder, control the lateral directional motion providing rolling and yawing moments correspondingly as a response to the corresponding control surfaces deflections.

The symbols used for the corresponding angles of rotation and control surface deflections are shown below:

| axis | $O_x$ | $O_y$ | $O_z$ |
|---|---|---|---|
| angle of rotation | $\phi$<br>(bank) | $\theta$<br>(pitch) | $\psi$<br>(yaw) |
| control surface defl. | $\xi$<br>(aileron) | $\eta$<br>(elevator) | $\zeta$<br>(rudder) |

In the longitudinal and lateral directional equations of motion, Equations 2-152 and 2-153, respectively, we have denoted by the symbol $\delta$ the control input due to control surfaces deflections.

The right-hand side of the longitudinal Equation 2-152 is expanded as follows:

$$
\begin{aligned}
X_\delta &= X_\eta \\
Z_\delta &= Z_\eta \\
M_\delta &= M_\eta + M_{\dot\eta}
\end{aligned}
\qquad (3\text{-}142)
$$

and the right-hand side of the lateral directional Equation 2-153 as:

$$\begin{aligned} Y_\delta &= Y_\zeta \\ L_\delta &= L_\xi + L_{\dot{\xi}} + L_\zeta \\ N_\delta &= N_\xi + N_\zeta + N_{\dot{\zeta}} \end{aligned} \tag{3-143}$$

In addition, to the right-hand side terms that take care of the main control surfaces, there may exist terms rising from the auxiliary surfaces such as flaps, stabilizers, dive brakes, etc.

A general approach suggests to examine the subscripted derivatives instead of the particular derivatives dedicated for a specific control surface deflection, i.e. the total effect by all the possible combinations of the control surface deflections. The approach is similar to that followed in Sections 3.2 and 3.3.

1. The $X_\delta$ Derivative

Is the rate of change in the x-force due to a control surface perturbation divided by the mass m.

$$X_\delta = \frac{1}{m} \cdot \frac{\partial X}{\partial \delta} \tag{3-144}$$

By $\delta$ we denote a general control surface deflection. For longitudinal control it might be elevator, stabilizer, flap, slat, dive brakes, etc.

According to Equation 3-27, the X-force is simplified to:

$$X = -\frac{1}{2} \rho V^2 S C_D + T \tag{3-145}$$

We can assume that neither V or T depends on the control deflection. Thus differentiating Equation 3-145, with respect to , we obtain:

$$\frac{\partial X}{\partial \delta} = -\frac{1}{2} \rho V^2 S \frac{\partial C_D}{\partial \delta} \qquad (3\text{-}146)$$

If we let V=Uo and

$$C_{D\delta} = \partial C_D / \partial \delta \qquad (3\text{-}147)$$

we arrive at the expression:

$$X_\delta = -\frac{\rho U_o^2 S}{2m} C_{D\delta} \qquad (3\text{-}148)$$

The quantity $C_{D\delta}$ represents a change in the drag coefficient due to a variation in a longitudinal control surface deflection.

A control deflection, positive or negative, increases the drag due to the greater projected frontal area, but this sometimes is desirable because it provides damping in an oscillatory perturbed motion.

2. The $Z_\delta$ Derivative

Is the rate of change in the Z-force due to a $\delta$ control surface perturbation divided by the mass m.

$$Z_\delta = \frac{1}{m} \cdot \frac{\partial Z}{\partial \delta} \qquad (3\text{-}149)$$

According to Equation 3-16, the Z-force is simplified to:

$$Z = W \cos \Theta - L \qquad (3\text{-}150)$$

or

$$Z = W \cos\Theta - \frac{1}{2}\rho V^2 S C_L \tag{3-151}$$

Neither W or $\theta$ depends on the $\delta$ control deflection. Thus differentiating Equation 3-151, with respect to $\delta$ we obtain:

$$\frac{\partial Z}{\partial \delta} = -\frac{1}{2}\rho V^2 S \frac{\partial C_L}{\partial \delta} \tag{3-152}$$

If we let V=Uo and

$$C_{L_\delta} = \partial C_L / \partial \delta \tag{3-153}$$

we arrive at the expression

$$Z_\delta = -\frac{\rho U_o^2 S}{2m} C_{L_\delta} \tag{3-154}$$

The quantity $C_{L_\delta}$ represents a change in the lift coefficient due to a variation in a longitudinal control surface deflection.

This derivative is a measure of the effectiveness of the elevator mainly, in producing a change in the lift force acting on the tailplane. A positive elevator deflection produces an increase of lift, i.e. negative Z-force.

It has a very small value on aircrafts with tail but a large value on tailless aircrafts.

3. The $M_\delta$ Derivative

Is the rate of change in the M-pitching moment due to a control surface perturbation divided by the Iyy moment of inertia

$$M_\delta = \frac{1}{I_{yy}} \cdot \frac{\partial M}{\partial \delta} \tag{3-155}$$

According to Equation 3-3, the pitching moment is expressed as:

$$M = \frac{1}{2} \rho V^2 S c C_M \qquad (3\text{-}156)$$

Differentiating Equation 3-156, with respect to $\delta$ we obtain:

$$\frac{\partial M}{\partial \delta} = \frac{1}{2} \rho V^2 S c \frac{\partial C_M}{\partial \delta} \qquad (3\text{-}157)$$

If we let V=Uo and

$$C_{M_\delta} = \partial C_M / \partial \delta \qquad (3\text{-}158)$$

we arrive at the expression

$$M_\delta = \frac{\rho U_o^2 S c}{2 I_{yy}} C_{M\delta} \qquad (3\text{-}159)$$

The quantity $C_{M\delta}$ represents a change in the pitching moment coefficient due to a variation in a longitudinal control surface deflection.

It is commonly referred to as the elevator power derivative because it is a measure of the elevator control effectiveness.

The change in Z-force produced by the elevator control surface deflection, multiplied by a rather long arm, provides a pitching moment in the negative sense, i.e. a positive elevator deflection results in a nose-down (negative) pitching moment.

That is the primary function of the elevator and is considered to be the most important of all the control

surface functions. It adjusts the angle of attack and it trims the vehicle in its longitudinal motion, by providing opposite pitching moments to accommodate disturbing moments and center of gravity relocations.

For a design point of view, the larger the center of gravity range is, the greater the elevator pitching moments must be provided, i.e. larger $C_{M_\delta}$ values.

4. The $Y_\delta$ Derivative

Is the change in the Y-side force due to a $\delta$ control surface perturbation divided by the mass, m.

$$Y_\delta = \frac{1}{m} \cdot \frac{\partial Y}{\partial \delta} \tag{3-160}$$

Differentiating Equation 3-75, with respect to $\delta$ we obtain:

$$\frac{\partial Y}{\partial \delta} = \frac{1}{2} \rho V^2 S \frac{\partial C_y}{\partial \delta} \tag{3-161}$$

If we let V=Uo and

$$C_{y_\delta} = \partial C_y / \partial \delta \tag{3-162}$$

we arrive at the expression:

$$Y_\delta = \frac{\rho U_o^2 S}{2m} C_{y_\delta} \tag{3-163}$$

The quantity $C_{y_\delta}$ represents the rate of change in the Y-side force coefficient due to a variation in a lateral control surface deflection.

This derivative rises mainly from the rudder. A positive rudder deflection is defined to produce a positive side force. Therefore, the value of this derivative is positive but small enough to be neglected.

Rudder deflections also cause an increase of the total drag of the vehicle but that has a negligible effect on the longitudinal motion, and is neglected.

### 5. The $L_\delta$ Derivative

Is the rate of change in the L-rolling moment due to a $\delta$ control surface perturbation, divided by the Ixx moment of inertia.

$$L_\delta = \frac{1}{I_{xx}} \cdot \frac{\partial L}{\partial \delta} \tag{3-164}$$

Differentiating Equation 3-76, with respect to $\delta$, we obtain:

$$\frac{\partial L}{\partial \delta} = \frac{1}{2} \rho V^2 S b \, \frac{\partial C_\ell}{\partial \delta} \tag{3-165}$$

If we let V=Uo and

$$C_{\ell_\delta} = \partial C_\ell / \partial \delta \tag{3-166}$$

we arrive at the expression:

$$L_\delta = \frac{\rho U_o^2 S b}{2 I_{xx}} \, C_{\ell_\delta} \tag{3-167}$$

The quantity $C_{\ell_\delta}$ represents the change in the $C_\ell$ rolling moment coefficient due to a variation in a lateral control surface deflection.

The side force generated by a rudder deflection, produces a rolling moment in the positive sense, i.e. a positive rudder deflection generates a positive rolling moment. In airplanes, with large vertical tails, this effect is negligible and is often neglected.

The main contribution to this derivative is from the ailerons, and it is a measure of the aileron effectiveness. A positive aileron deflection is defined as the one to produce a positive rolling moment, i.e. left aileron down and right aileron up, therefore, the value of this derivative is positive.

In lateral-directional dynamics, this derivative is probably the most important because it establishes the maximum available rate of roll of an airplane which is a very important consideration in combat environment.

It also plays an important role in take-offs, and landings, where adequate and rapid lateral control is highly desirable.

Rapid rate of change of aileron deflections gives also a rise to this derivative due to lag effects, i.e. not immediate adjustment of the airstream.

### 6. The $N_\delta$ Derivative

Is the rate of change in the N-yawing moment due to a $\delta$ control surface perturbation, divided by the Izz moment of inertia.

$$N_\delta = \frac{1}{I_{zz}} \cdot \frac{\partial N}{\partial \delta} \qquad (3\text{-}168)$$

Differentiating Equation 3-77, with respect to $\delta$, we obtain:

$$\frac{\partial N}{\partial \delta} = \frac{1}{2} \rho V^2 S c \frac{\partial C_n}{\partial \delta} \qquad (3\text{-}169)$$

If we let V=Uo

$$C_{n_\delta} = \partial C_n / \partial \delta \qquad (3\text{-}170)$$

we arrive at the expression:

$$N_\delta = \frac{\rho U_o^2 S c}{2 I_{zz}} C_{n_\delta} \qquad (3\text{-}171)$$

The quantity $C_{n_\delta}$ represents the change in the $C_n$ yawing moment coefficient due to a variation in a lateral control surface deflection.

The side force generated by a rudder deflection, produces a yawing moment in the negative sense, i.e. a positive or to the left rudder deflection, produces a negative yawing moment. This derivative is a measure of the rudder effectiveness.

Rapid rate of change of rudder deflections gives also a rise to this derivative due to lag effects.

This derivative plays an important role in crosswind takeoffs, and landings, and counteracts adverse yaw in rolling maneuvers.

Finally, aileron deflections give also a rise to this derivative since each aileron introduces a drag force of different magnitude. This effect may be positive or negative depanding on the rigging of the aileron and the angle of attack of the airframe.

## 3.5 DETERMINATION OF THE STABILITY DERIVATIVES AND OTHER CONSIDERATIONS

All the stability derivatives discussed so far rise from the flight vehicle's geometry, being highly dependent upon the design layout. We have referred mostly in this chapter to the conventional airplane with horizontal tail and we have seen that the derivatives are made up of a number of separate contributions.

We had strictly remained in the dimensional form of the stability derivatives and we simply expressed them in terms of airstream characteristics, airframe geometry and non-dimensional derivatives.

Dimensional form has the advantage of obtaining real time solutions rather than working with normalized time mass and inertia components that must also be non-dimensional in considering that form.

The advantage of non-dimensional derivatives, i.e. independent of size and airstream characteristics is maintained because dimensional derivatives are expressed with

the corresponding non-dimensionals, some of which though do depend upon the airstream and airframe characteristics.

Thus, some of the dimensional expressions and some of the non-dimensionals are constants, but they can be expressed in polynomial form.

As an example, for the METEOR 7:

$$C_{\ell_v} = -0.052 - 0.009\ C_L$$

$$C_{\ell_r} = 0.0295 + 0.159\ C_L$$

$$C_{n_p} = -0.042\ C_L$$

$$C_{n_p} = -0.182 - 0.052\ C_L^2$$

For others, plots may exist indicating their variation with respect to $C_L$ or mach number.

During the development of an aircraft, it is important to know the numerical values of the derivatives under various conditions of flight. In the first stage, theoretical estimates can be made supported by wind tunnel tests and model experiments.

By then, simulation in digital computers can be made to obtain response characteristics to initial disturbances or control deflections and determine whether the values of the derivatives are satisfactory.

When the aircraft has been built, flight tests are carried out to confirm the theoretical with the actual responses.

There also exist flight test techniques for determining the values of the derivatives.

Tables 3.1 and 3.2 summarize the longitudinal and lateral directional derivatives including definition, unit, expression and auxilliary relations.

## TABLE 3-1

## LONGITUDINAL STABILITY DERIVATIVES

| Quantity | Definition | Unit | Evaluation | Aux. Relation |
|---|---|---|---|---|
| $X_u$ | $\frac{1}{m} \cdot \frac{\partial X}{\partial u}$ | 1/sec | $-\frac{\rho U_0 S}{m}\left[C_D + C_{D_u}\right]$ | $C_{D_u} = \frac{U_0}{2} \frac{\partial C_D}{\partial u}$ |
| $Z_u$ | $\frac{1}{m} \cdot \frac{\partial Z}{\partial u}$ | 1/sec | $-\frac{\rho U_0 S}{m}\left[C_L + C_{L_u}\right]$ | $C_{L_u} = \frac{U_0}{2} \frac{\partial C_L}{\partial u}$ |
| $M_u$ | $\frac{1}{I_{yy}} \cdot \frac{\partial M}{\partial u}$ | 1/ft · sec | $\frac{\rho U_0 S c}{2 I_{yy}}\left[C_M + C_{M_u}\right]$ | $C_{M_u} = \frac{U_0}{2} \frac{\partial C_M}{\partial u}$ |
| $X_w$ | $\frac{1}{m} \cdot \frac{\partial X}{\partial w}$ | 1/sec | $\frac{\rho U_0 S}{2m}\left[C_L - C_{D_\alpha}\right]$ | $C_{D_\alpha} = \frac{\partial C_D}{\partial \alpha}$ |
| $Z_w$ | $\frac{1}{m} \cdot \frac{\partial Z}{\partial w}$ | 1/sec | $-\frac{\rho U_0 S}{2m}\left[C_{L_\alpha} + C_D\right]$ | $C_{L_\alpha} = \frac{\partial C_L}{\partial \alpha}$ |
| $M_w$ | $\frac{1}{I_{yy}} \cdot \frac{\partial M}{\partial w}$ | 1/ft-sec | $\frac{\rho U_0 S c}{2 I_{yy}} C_{M_\alpha}$ | $C_{M_\alpha} = \frac{\partial C_M}{\partial \alpha}$ |
| $Z_{\dot{w}}$ | $\frac{1}{m} \cdot \frac{\partial Z}{\partial \dot{w}}$ | 1/1 | $-\frac{\rho S c}{4m} C_{L_{\dot{\alpha}}}$ | $C_{L_{\dot{\alpha}}} = \partial C_L / \partial(\dot{\alpha} c / 2U_0)$ |
| $M_{\dot{w}}$ | $\frac{1}{I_{yy}} \cdot \frac{\partial M}{\partial \dot{w}}$ | 1/ft | $\frac{\rho S c^2}{4 I_{yy}} C_{M_{\dot{\alpha}}}$ | $C_{M_{\dot{\alpha}}} = \partial C_M / \partial(\dot{\alpha} c / 2U_0)$ |
| $Z_q$ | $\frac{1}{m} \cdot \frac{\partial Z}{\partial q}$ | ft/rad sec | $-\frac{\rho U_0 S c}{4m} C_{L_q}$ | $C_{L_q} = \partial C_L / \partial(q c / 2U_0)$ |
| $M_q$ | $\frac{1}{I_{yy}} \cdot \frac{\partial M}{\partial q}$ | 1/sec | $\frac{\rho U_0 S c^2}{4 I_{yy}} C_{M_q}$ | $C_{M_q} = \partial C_M / \partial(q c / 2U_0)$ |
| $X_\delta$ | $\frac{1}{m} \cdot \frac{\partial X}{\partial \delta}$ | ft/rad-sec$^2$ | $-\frac{\rho U_0^2 S}{2m} C_{D_\delta}$ | $C_{D_\delta} = \frac{\partial C_D}{\partial \delta}$ |
| $Z_\delta$ | $\frac{1}{m} \cdot \frac{\partial Z}{\partial \delta}$ | ft/rad-sec$^2$ | $-\frac{\rho U_0^2 S}{2m} C_{L_\delta}$ | $C_{L_\delta} = \frac{\partial C_L}{\partial \delta}$ |
| $M_\delta$ | $\frac{1}{I_{yy}} \cdot \frac{\partial M}{\partial \delta}$ | 1/rad sec$^2$ | $\frac{\rho U_0^2 S c}{2 I_{yy}} C_{M_\delta}$ | $C_{M_\delta} = \frac{\partial C_M}{\partial \delta}$ |

## TABLE 3-2

## LATERAL DIRECTIONAL STABILITY DERIVATIVES

| Quantity | Definition | Unit | Evaluation | Aux. Relation |
|---|---|---|---|---|
| $Y_v$ | $\frac{1}{mU_0}\cdot\frac{\partial Y}{\partial \beta}$ | 1/sec | $\frac{\rho U_0 S}{2m} C_{y_\beta}$ | $C_{y_\beta} = \frac{\partial C_y}{\partial \beta}$ |
| $L_v$ | $\frac{1}{U_0 I_{xx}}\cdot\frac{\partial L}{\partial \beta}$ | 1/ft-sec | $\frac{\rho U_0 S b}{2 I_{xx}} C_{l_\beta}$ | $C_{l_\beta} = \frac{\partial C_l}{\partial \beta}$ |
| $N_v$ | $\frac{1}{U_0 I_{zz}}\cdot\frac{\partial N}{\partial \beta}$ | 1/ft-sec | $\frac{\rho U_0 S b}{2 I_{zz}} C_{n_\beta}$ | $C_{n_\beta} = \frac{\partial C_n}{\partial \beta}$ |
| $Y_{\dot v}$ | $\frac{1}{mU_0}\cdot\frac{\partial Y}{\partial \dot\beta}$ | 1/1 | $\frac{\rho S b}{4m} C_{y_{\dot\beta}}$ | $C_{y_{\dot\beta}} = \partial C_y / \partial(\dot\beta b/2U_0)$ |
| $L_{\dot v}$ | $\frac{1}{U_0 I_{xx}}\cdot\frac{\partial L}{\partial \dot\beta}$ | 1/ft | $\frac{\rho S b^2}{4 I_{xx}} C_{l_{\dot\beta}}$ | $C_{l_{\dot\beta}} = \partial C_l / \partial(\dot\beta b/2U_0)$ |
| $N_{\dot v}$ | $\frac{1}{U_0 I_{zz}}\cdot\frac{\partial N}{\partial \dot\beta}$ | 1/ft | $\frac{\rho S b^2}{4 I_{zz}} C_{n_{\dot\beta}}$ | $C_{n_{\dot\beta}} = \partial C_n / \partial(\dot\beta b/2U_0)$ |
| $Y_p$ | $\frac{1}{m}\cdot\frac{\partial Y}{\partial p}$ | ft/rad sec | $\frac{\rho U_0 S b}{4m} C_{y_p}$ | $C_{y_p} = \partial C_y / \partial(pb/2U_0)$ |
| $L_p$ | $\frac{1}{I_{xx}}\cdot\frac{\partial L}{\partial p}$ | 1/sec | $\frac{\rho U_0 S b^2}{4 I_{xx}} C_{l_p}$ | $C_{l_p} = \partial C_l / \partial(pb/2U_0)$ |
| $N_p$ | $\frac{1}{I_{zz}}\cdot\frac{\partial N}{\partial p}$ | 1/sec | $\frac{\rho U_0 S b^2}{4 I_{zz}} C_{n_p}$ | $C_{n_p} = \partial C_n / \partial(pb/2U_0)$ |
| $Y_r$ | $\frac{1}{m}\cdot\frac{\partial Y}{\partial r}$ | ft/rad sec | $\frac{\rho U_0 S b}{4m} C_{y_r}$ | $C_{y_r} = \partial C_y / \partial(rb/2U_0)$ |
| $L_r$ | $\frac{1}{I_{xx}}\cdot\frac{\partial L}{\partial r}$ | 1/sec | $\frac{\rho U_0 S b^2}{4 I_{xx}} C_{l_r}$ | $C_{l_r} = \partial C_l / \partial(rb/2U_0)$ |
| $N_r$ | $\frac{1}{I_{zz}}\cdot\frac{\partial N}{\partial r}$ | 1/sec | $\frac{\rho U_0 S b^2}{4 I_{zz}} C_{n_r}$ | $C_{n_r} = \partial C_n / \partial(rb/2U_0)$ |
| $Y_\delta$ | $\frac{1}{m}\cdot\frac{\partial Y}{\partial \delta}$ | ft/rad sec$^2$ | $\frac{\rho U_0^2 S}{2m} C_{y_\delta}$ | $C_{y_\delta} = \frac{\partial C_y}{\partial \delta}$ |
| $L_\delta$ | $\frac{1}{I_{xx}}\cdot\frac{\partial L}{\partial \delta}$ | 1/rad sec$^2$ | $\frac{\rho U_0^2 S b}{2 I_{xx}} C_{l_\delta}$ | $C_{l_\delta} = \frac{\partial C_l}{\partial \delta}$ |
| $N_\delta$ | $\frac{1}{I_{zz}}\cdot\frac{\partial N}{\partial \delta}$ | 1/rad sec$^2$ | $\frac{\rho U_0^2 S' c}{2 I_{zz}} C_{n_\delta}$ | $C_{n_\delta} = \frac{\partial C_n}{\partial \delta}$ |

# CHAPTER 4

# LONGITUDINAL DYNAMICS

## 4.1 INTRODUCTION

Having specified all the vehicle's dynamic properties, we will now examine the dynamic motion, i.e. the vehicle's response to disturbances and control inputs.

Since any flight vehicle is a dynamic system, described by differential equations of time, we must expect in a particular situation, oscillatory or exponential converging or diverging response.

Any particular response will be associated by its corresponding characteristic root location in the complex plane, as we have clearly seen in the dynamic response of the spring mass damper system.

The characteristic polynomials in the Laplace variable (s) involved, are usually of the third degree or higher and thus hard to solve by hand.

In the past, a lot of approximations were introduced and even reduction of airframe's degrees of freedom technique was required to analyze coarsely a particular mode of motion, by reducing in effect the degree of the characteristic polynomial.

In this chapter, we will analyze the longitudinal motion of a conventional airplane. We will first discuss the

Laplace Transform solution by obtaining the roots of the associated characteristic polynomial. Then we will apply the reduction of degrees of freedom technique to isolate and analyze a particular mode of motion. Finally, the equations of motion will be rearranged in a form that is more suitable for computer solutions.

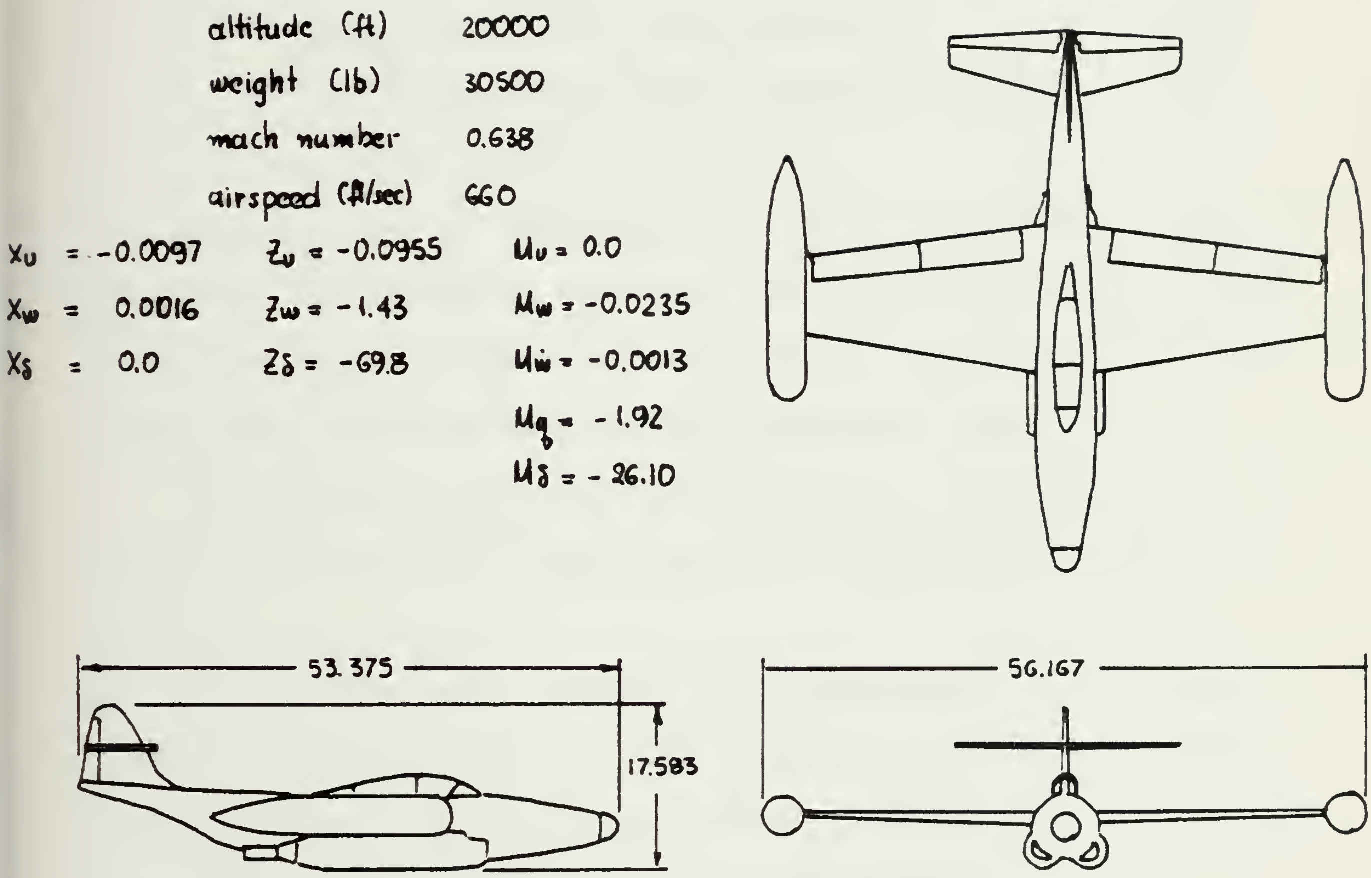
As an example, the longitudinal dynamic motion of the aircraft shown in Figure 4-1 will be analyzed with the parameters indicated.

## 4.2 CHARACTERISTIC POLYNOMIAL OF LONGITUDINAL MOTION

Longitudinal equations of motion are farther simplified by neglecting the effect of the $Z_q$ and $Z_{\dot{w}}$ derivatives. Experimental results have shown that the effect of neglecting them is only of secondary importance. Further a straight level flight is assumed as the reference flight condition (i.e. $\gamma_0 = 0$), hence, Equation 2-152 can be reduced to:

$$\begin{vmatrix} s - X_\upsilon & -X_w & 0 & g \\ -Z_\upsilon & s - Z_w & -U_0 & 0 \\ -M_\upsilon & -sM_{\dot{w}} - M_w & s - M_q & 0 \\ 0 & 0 & 1 & -s \end{vmatrix} \cdot \begin{vmatrix} \upsilon \\ w \\ q \\ \theta \end{vmatrix} = \begin{vmatrix} X_\delta \\ Z_\delta \\ M_\delta \\ 0 \end{vmatrix} \delta \qquad (4\text{-}1)$$

The characteristic polynomial is found by evaluation of the above determinant. The Equation 4-1 can be reduced in 3 equations of the original $\upsilon$,w,q, longitudinal variables by avoiding to write down the definition equation for q, but



| | |
|---|---|
| altitude (ft) | 20000 |
| weight (lb) | 30500 |
| mach number | 0.638 |
| airspeed (ft/sec) | 660 |

$X_u = -0.0097$ $Z_u = -0.0955$ $M_u = 0.0$

$X_w = 0.0016$ $Z_w = -1.43$ $M_w = -0.0235$

$X_\delta = 0.0$ $Z_\delta = -69.8$ $M_{\dot{w}} = -0.0013$

$M_q = -1.92$

$M_\delta = -26.10$

(all dimensions in ft)

Figure 4-1

Conventional Airplane and Parameters
Used for the Numerical Example

that doesn't simplify the hand calculations. The determinant comes out to be:

$$As^4 + Bs^3 + Cs^2 + Ds + E = 0 \tag{4-2}$$

where

$$A = 1 \tag{4-3}$$

$$B = -M_q - U_o M_{\dot{w}} - Z_w - X_u \tag{4-4}$$

$$C = Z_w M_q - U_o M_w - X_w Z_u + X_u (M_q + U_o M_{\dot{w}} + Z_w) \tag{4-5}$$

$$D = -X_u (Z_w M_q - U_o M_w) + Z_u (X_w M_q + g M_{\dot{w}}) - M_u (U_o X_w - g) \tag{4-6}$$

$$E = g (Z_u M_w - M_u Z_w) \tag{4-7}$$

Equation 4-2 is referred to as the longitudinal stability quartic. The roots in this case for any conventional vehicle come out to be complex with negative real parts, which are associated with two oscillatory convergent motions.

The overall longitudinal motion of the vehicle is a superposition or a combination of these two oscillatory modes.

One of the modes usually is associated with a pair of complex roots located near the origin of the negative complex plane. This mode is called "Phugoid motion" and is a fairly lightly damped oscillation with long periodic time.

The other mode usually is associated with a pair of complex roots located far from the origin of the negative complex plane. This mode is called "Short period motion" and is a heavily damped oscillation with very short periodic time.

The longitudinal oscillatory modes are shown in Figure 4-2.

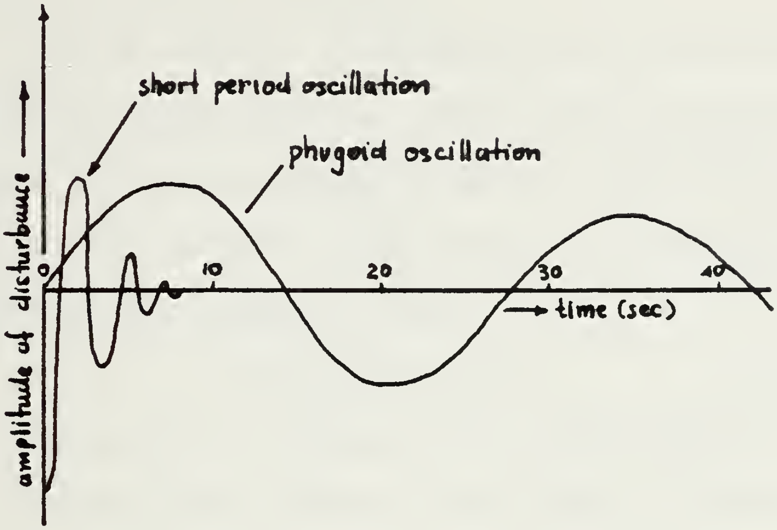


Figure 4-2

The Longitudinal Oscillatory Modes

For the given example we find

$$A = 1.0 \tag{4-8}$$

$$B = 4.2177 \tag{4-9}$$

$$C = 18.2962 \tag{4-10}$$

$$D = 0.1814 \tag{4-11}$$

$$E = 0.0722 \tag{4-12}$$

and the characteristic polynomial is

$$s^4 + 4.2177\, s^3 + 18.2962\, s^2 + 0.1814\, s + 0.0722 = 0 \tag{4-13}$$

which has roots

$$-0.0045 \pm j\, 0.0627 \tag{4-14}$$

$$-2.1043 \pm j\, 3.7184 \tag{4-15}$$

The pair of roots -0.0045 ± j 0.0627 correspond to the phugoid motion and the pair of roots -2.1043 ± j 3.7184 correspond to the short motion.

The fourth degree polynomial can be factored in two second degree polynomials. Each one of the second degree polynomials is associated with the corresponding mode.

Further it is more convenient to put each one of the polynomials in the familiar form

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \tag{4-16}$$

so that the natural frequency $\omega_n$, the damping $\zeta$ are figured out by inspection while the damped frequency can be determined by:

$$\omega_d = \omega_n (1-\zeta^2)^{1/2} \tag{4-17}$$

If we denote with the subscript (SP) the short period quantities and with the subscript (PH) the phugoid quantities, the characteristic polynomial may be written in the form:

$$(s^2 + 2\zeta_{SP}\omega_{n_{SP}} s + \omega_{n_{SP}}^2)(s^2 + 2\zeta_{PH}\omega_{n_{PH}} s + \omega_{n_{PH}}^2) = 0 \tag{4-18}$$

Those quantities and their relation is shown in the complex plane in Figure 4-3.

The roots of the characteristic polynomial are clearly shown in Figure 4-3 to be

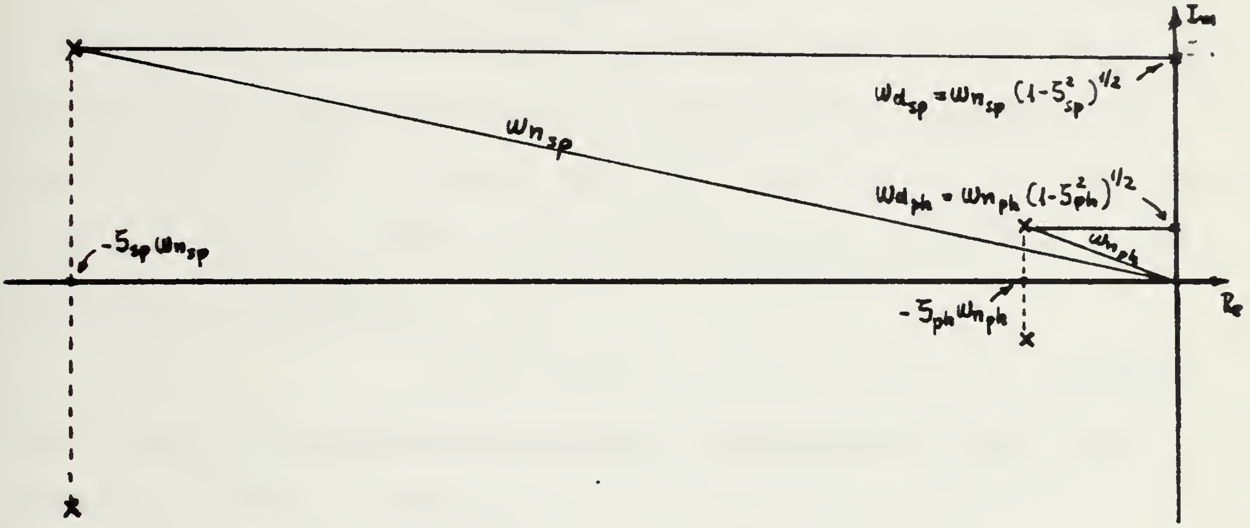
$$-\zeta_{PH}\omega_{n_{PH}} \pm j\,\omega_{d_{PH}} \tag{4-19}$$

$$-\zeta_{SP}\omega_{n_{SP}} \pm j\,\omega_{d_{SP}} \tag{4-20}$$

or

$$-\zeta_{PH}\,\omega_{n_{PH}} \pm j\,\omega_{n_{PH}}\left(1-\zeta_{PH}^2\right)^{1/2} \tag{4-21}$$

$$-\zeta_{SP}\,\omega_{n_{SP}} \pm j\,\omega_{n_{SP}}\left(1-\zeta_{SP}^2\right)^{1/2} \tag{4-22}$$



$I_m$
$\omega_{d_{sp}} = \omega_{n_{sp}}(1-\zeta_{sp}^2)^{1/2}$
$\omega_{n_{sp}}$
$\omega_{d_{ph}} = \omega_{n_{ph}}(1-\zeta_{ph}^2)^{1/2}$
$\omega_{n_{ph}}$
$-\zeta_{sp}\omega_{n_{sp}}$
$-\zeta_{ph}\omega_{n_{ph}}$
$R_e$

Figure 4-3

Longitudinal Roots and Characteristic Quantities in the Complex Plane

Other quantities of importance in the longitudinal dynamics are: The phugoid period denoted by $T_{PH}$, the short period period $T_{SP}$ given by:

$$T_{PH} = 2\pi/\omega_{d_{PH}} \quad \text{(sec)} \tag{4-23}$$

$$T_{SP} = 2\pi/\omega_{d_{SP}} \quad \text{(sec)} \tag{4-24}$$

and the time for half amplitude for phugoid and short period given by:

$$t_{1/2_{PH}} = \frac{0.6931}{\zeta_{PH}\,\omega_{n_{PH}}} \qquad (4\text{-}25)$$

$$t_{1/2_{SP}} = \frac{0.6931}{\zeta_{SP}\,\omega_{n_{SP}}} \qquad (4\text{-}26)$$

The Equations 4-25 and 4-26 come from the fact that the oscillatory response decays exponentially described by the envelope $e^{-Ret}$ where Re is the real part of the characteristic root. Therefore, to obtain half of the amplitude the relation

$$e^{-Ret} = 0.5 \qquad (4\text{-}27)$$

must exist. By taking the natural logarithm of both sides we obtain the above relations.

For our example, the characteristic polynomial of Equation 4-13 can be factored in the form

$$(s^2 + 0.0090\ s + 0.00395)(s^2 + 4.2086s + 18.2546) \qquad (4\text{-}28)$$

From this, by inspection we can determine:

$$\omega_{n_{PH}} = (0.00395)^{1/2} = 0.0628 \text{ rad/sec} \qquad (4\text{-}29)$$

$$\omega_{n_{SP}} = (18.2546)^{1/2} = 4.2725 \text{ rad/sec} \qquad (4\text{-}30)$$

$$\zeta_{PH} = \frac{0.0090}{2 \times 0.0628} = 0.0717 \qquad (4\text{-}31)$$

$$\zeta_{SP} = \frac{4.2086}{2 \times 4.2725} = 0.4925 \qquad (4\text{-}32)$$

and also

$$t_{1/2_{PH}} = \frac{0.6931}{0.0717 \cdot 0.0628} = 153.93 \text{ sec} \qquad (4\text{-}33)$$

$$t_{1/2\,SP} = \frac{0.6931}{0.4925 \times 4.2725} = 0.33 \text{ sec} \quad (4\text{-}34)$$

$$\omega_{d_{PH}} = 0.0628(1-0.0717^2)^{1/2} = 0.0627 \text{ rad/sec} \quad (4\text{-}35)$$

$$\omega_{d_{SP}} = 4.2725(1-0.4925^2)^{1/2} = 3.7184 \text{ rad/sec} \quad (4\text{-}36)$$

$$T_{PH} = \frac{2\pi}{0.0627} = 100.21 \text{ sec} \quad (4\text{-}37)$$

$$T_{SP} = \frac{2\pi}{3.7184} = 1.69 \text{ sec} \quad (4\text{-}38)$$

It is worthwhile to note that:

(a) The damped frequency is indicated by the imaginary part of the corresponding characteristic root.

(b) The real part of the characteristic root indicates the exponentially decaying envelope of the corresponding oscillatory motion, i.e. the damping.

(c) The following relations hold

$$B = 2\zeta_{PH}\,\omega_{n_{PH}} + 2\zeta_{SP}\,\omega_{n_{SP}} \quad (4\text{-}39)$$

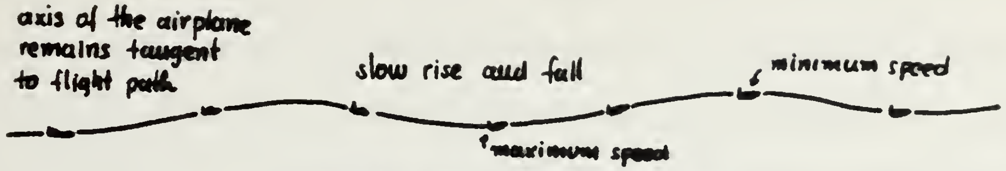
$$C = \omega_{n_{PH}}^2 + 4\zeta_{PH}\,\omega_{n_{PH}}\,\zeta_{SP}\,\omega_{n_{SP}} + \omega_{n_{SP}}^2 \quad (4\text{-}40)$$

$$D = 2\zeta_{PH}\,\omega_{n_{PH}}\,\omega_{n_{SP}}^2 + 2\zeta_{SP}\,\omega_{n_{SP}}\,\omega_{n_{PH}}^2 \quad (4\text{-}41)$$

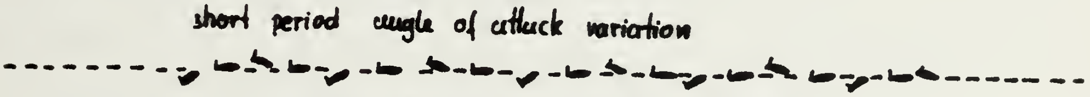
$$E = \omega_{n_{PH}}^2\,\omega_{n_{SP}}^2 \quad (4\text{-}42)$$

The two types of dynamical longitudinal oscillations are shown in Figure 4-4.

The forward velocity $\upsilon$ and the pitch angle $\theta$ are usually excited in the phugoid mode while the w velocity and, hence, the angle of attack $\alpha$ remains nearly unchanged.



Phugoid longitudinal oscillation



Short-period longitudinal oscillation

Figure 4-4

Dynamic Longitudinal Oscillations

In contrast, the angle of attack is heavily excited in the short period mode as well as the pitch angle $\theta$, while the forward velocity remains nearly unchanged in that mode.

The pilot can generally control the phugoid oscillation although because of his slow reaction it is possible to worsen to unstable due to out of phase action of his control. Phugoid oscillation must be highly considered in take-offs and landings. Short period is usually out of pilot control, but is of importance in automatic control.

The requirements that an aircraft should satisfy are written in appropriate manuals. It is usually imposed that the oscillation should not take more than one cycle to damp to half amplitude.

Flight test can be carried out to determine the oscillations characteristics under different conditions of flight. Gyroscopic instruments can be placed and recordings of the longitudinal variables can be made possible to obtain. The aircraft is trimmed to the selected speed and height and longitudinal oscillation is initiated by an abrupt elevator deflection.

## 4.3 PHUGOID AND SHORT PERIOD APPROXIMATIONS

The fact that the angle of attack and hence, the w-velocity remains nearly unchanged in the phugoid mode, gave the idea of determining the phugoid quadratic by:

(a) eliminating the w terms.

(b) eliminating the pitching moment equation since no considerable pitching moment variation occurs, and

(c) letting $q=s\theta$, obtaining thus the determinant:

$$\begin{vmatrix} s-X_u & g \\ -Z_u & -U_0 s \end{vmatrix} \tag{4-43}$$

The characteristic phugoid quadratic comes out to be:

$$s^2 - X_u s - \frac{g}{U_0} Z_u = 0 \tag{4-44}$$

For example, given, the characteristic quadratic is

$$s^2 + 0.0097s + 0.0047 = 0 \tag{4-45}$$

which has roots

$$-0.0049 \pm j\,0.0681 \tag{4-46}$$

These roots are approximately equal to the exact phugoid roots.

From Equation 4-44 it is clear that the natural frequency of the phugoid is heavily dependent upon the $Z_u$ derivative while the damping of the phugoid is dependent upon the $X_u$ derivative. It is also obvious that high speeds tend to reduce the phugoid frequency and to increase the corresponding period.

Since for subsonic straight level flight the $Z_u$ derivative is simply expressed as:

$$Z_u = -2g / U_o \qquad (4\text{-}47)$$

the following approximate relations hold for the phugoid

$$\omega_{n_{PH}} \doteq \sqrt{2} \, \frac{g}{U_o} \qquad (4\text{-}48)$$

$$\zeta_{PH} = - \frac{U_o X_u}{2\sqrt{2} \, g} \qquad (4\text{-}49)$$

Aircraft designers, in order to achieve larger values of phugoid damping, must increase the value of the $X_u$ derivative but this derivative is proportional to the drag. Fortunately, in take-offs and landings where phugoid motion is undesired, high drag devices such as flaps are used anyway.

A more precise approximation for the phugoid quadratic is given by leaving the w terms and the $M_u$ term which in modern crafts is never zero. This is a three degree of freedom phugoid approximation.

$$\begin{vmatrix} s-X_u & X_w & g \\ -Z_u & s-Z_w & -U_0 \\ -M_u & -M_w & 0 \end{vmatrix} \tag{4-50}$$

The characteristic phugoid quadratic becomes then

$$s^2 + \left[-X_u + \frac{M_u(U_0X_w-g)}{U_0M_w}\right]s - \frac{g}{U_0}\left(Z_u - \frac{M_u}{M_w}Z_w\right) \tag{4-51}$$

which gives

$$\omega_{n_{PH}} \doteq \left[\frac{g}{U_0}\left(Z_u - \frac{M_u}{M_w}Z_w\right)\right]^{1/2} \tag{4-52}$$

and

$$\zeta_{PH} \doteq \frac{\left[-X_u + \dfrac{M_u\,(U_0X_w-g)}{U_0M_w}\right]}{2\left[\dfrac{g}{U_0}\left(Z_u - \dfrac{M_u}{M_w}Z_w\right)\right]^{1/2}} \tag{4-53}$$

Short period approximations can be made possible by:

(a) eliminating the v-equations since very small v-velocity change occurs.

(b) setting all X-force derivatives to zero since very small X-force variation occurs.

(c) letting $q=s\theta$, obtaining thus the determinant:

$$\begin{vmatrix} s-Z_w & -U_0 \\ -sM_{\dot{w}}-M_w & s-M_q \end{vmatrix} \tag{4-54}$$

The characteristic short period quadratic comes out to be:

$$s^2 + (-U_0M_{\dot{w}} - M_q - Z_w)s + (-U_0M_w + M_qZ_w) = 0 \tag{4-55}$$

For the example given, the characteristic quadratic is:

$$s^2 + 4.208s + 18.2556 = 0 \qquad (4\text{-}56)$$

which has roots

$$-2.104 \pm j\,3.7187 \qquad (4\text{-}57)$$

These roots are approximately equal to the exact short period roots.

In summary, for typical flight conditions, the phugoid mode can be considered to consist of changes in the $\upsilon$-velocity and the $\theta$-pitch angle, while the short period mode can be considered to consist of changes in the w-velocity (angle of attack) and the pitch rate q.

For the short period mode, the following approximate relations hold:

$$\omega_{n_{SP}} = (M_q Z_w - U_o M_w)^{1/2} \qquad (4\text{-}58)$$

$$\zeta_{SP} = \frac{-(U_o M_{\dot{w}} + M_q + Z_w)}{2(M_q Z_w - U_o M_w)^{1/2}} \qquad (4\text{-}59)$$

A critical condition occurs when the center of gravity is in that position for which

$$M_q Z_w = U_o M_w \qquad (4\text{-}60)$$

For this case the characteristic equation becomes:

$$s\left[s + (-U_o M_{\dot{w}} - M_q - Z_w)\right] = 0 \qquad (4\text{-}61)$$

which has roots

$$0 \,, \qquad U_o M_{\dot{w}} + M_q + Z_w \qquad (4\text{-}62)$$

The non-zero root is usually negative which yields in a non-oscillatory and heavily damped exponential response.

## 4.4 SOLUTION TO THE LONGITUDINAL EQUATIONS

The longitudinal equations form a set of linear first order differential equations with constant coefficients. They are accompanied by a set of equal numbered initial conditions, i.e. initial deviations from the equilibrium flight conditions, and/or a number of control inputs.

Many ways have been developed for solving sets of differential equations. Laplace Transform method is usually the simplest although it is not suitable for computer solutions.

In the development of the solution, we would like to rearrange the equations in the matrix form:

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) \qquad (4\text{-}63)$$

where $\underline{x}(t)$ is a n-dimensional time varying column vector named state vector, $\underline{u}(t)$ is a r-dimensional time varying column vector named control input vector, $\underline{A}$ is a nxn constant matrix and $\underline{B}$ is a nxr constant matrix.

Taking the Laplace Transform of Equation 4-63 according to theorem 3 of Appendix A, we obtain:

$$s\,\underline{X}(s) = \underline{x}(0) + \underline{A}\,\underline{X}(s) + \underline{B}\,\underline{U}(s) \qquad (4\text{-}64)$$

or

$$(s\underline{I} - \underline{A})\,\underline{X}(s) = \underline{x}(0) + \underline{B}\,\underline{U}(s) \qquad (4\text{-}65)$$

or

$$\underline{X}(s) = (s\underline{I} - \underline{A})^{-1}\,\underline{x}(0) + (s\underline{I} - \underline{A})^{-1}\,\underline{B}\,\underline{U}(s) \qquad (4\text{-}66)$$

where $\underline{I}$ is the n x n identity matrix and $\underline{x}(0)$ is the initial condition column vector.

The first term of the transformed solution corresponds to the initial conditions and is the homogeneous solution and the second term corresponds to the control input and is the particular solution.

By inverse Laplace Transforming Equation 4-66 we obtain the time solution:

$$\underline{x}(t) = \left[\mathcal{L}^{-1}(s\underline{I}-\underline{A})^{-1}\right]\underline{x}(0) + \mathcal{L}^{-1}\left[(s\underline{I}-\underline{A})^{-1}\underline{B}\,\underline{U}(s)\right] \qquad (4\text{-}67)$$

To implement this solution, in a digital computer, time domain formulation is followed.

The key is to multiply the matrix differential Equation 4-63 by exp(-$\underline{A}$t) obtaining thus:

$$e^{-\underline{A}t}\left[\dot{\underline{x}}(t)-\underline{A}\,\underline{x}(t)\right] = e^{-\underline{A}t}\underline{B}\,\underline{u}(t) \qquad (4\text{-}68)$$

or

$$\frac{d}{dt}\left[e^{-\underline{A}t}\,\underline{x}(t)\right] = e^{-\underline{A}t}\underline{B}\,\underline{u}(t) \qquad (4\text{-}69)$$

Integrating between 0 and t, Equation 4-69 gives

$$e^{-\underline{A}t}\,\underline{x}(t) = \underline{x}(0) + \int_0^t e^{-\underline{A}\tau}\underline{B}\,\underline{u}(\tau)\,d\tau \qquad (4\text{-}70)$$

or

$$\underline{x}(t) = e^{\underline{A}t}\,\underline{x}(0) + \int_0^t e^{\underline{A}(t-\tau)}\underline{B}\,\underline{u}(\tau)\,d\tau \qquad (4\text{-}71)$$

The exponential term is expanded in Taylor series as:

$$e^{\underline{A}t} = \underline{I} + \underline{A}t + \frac{1}{2!}\underline{A}^2t^2 + \cdots + \frac{1}{k!}\underline{A}^k t^k \qquad (4\text{-}72)$$

In reality, a series truncated to ten or twenty terms is often quite sufficient for normal accuracy requirements.

The computer program determines the values of the state variables at time intervals selected by the user.

We now go back to the untransformed longitudinal equations and try to arrange them in the form of the matrix Equation 4-63.

For convenience, the untransformed equations are replicated below:

$$\dot{u} - (X_u)u - (X_w)w + g\theta = (X_\delta)\delta \qquad (4\text{-}73)$$

$$-(Z_u)u + \dot{w} - (Z_w)w - (U_0)q = (Z_\delta)\delta \qquad (4\text{-}74)$$

$$-(M_u)u - (M_{\dot{w}})\dot{w} - (M_w)w + \dot{q} - (M_q)q = (M_\delta)\delta \qquad (4\text{-}75)$$

$$\dot{\theta} = q \qquad (4\text{-}76)$$

Equation 4-73 is written:

$$\dot{u} = (X_u)u + (X_w)w - (g)\theta + (X_\delta)\delta \qquad (4\text{-}77)$$

Equation 4-74 is written:

$$\dot{w} = (Z_u)u + (Z_w)w + (U_0)q + (Z_\delta)\delta \qquad (4\text{-}78)$$

Equation 4-75 is written:

$$\dot{q} = (M_u)u + (M_w)w + (M_{\dot{w}})\dot{w} + (M_q)q + (M_\delta)\delta \qquad (4\text{-}79)$$

or substituting Equation 4-78 for $\dot{w}$ we obtain:

$$\dot{q} = (M_u + M_{\dot{w}}Z_u)u + (M_w + M_{\dot{w}}Z_w)w + (M_{\dot{w}}U_0)q + (M_\delta + M_{\dot{w}}Z_\delta)\delta \qquad (4\text{-}80)$$

or if we define:

$$M_u^* = M_u + M_{\dot{w}}Z_u \qquad (4\text{-}81)$$

$$M_w^* = M_w + M_{\dot{w}}Z_w \qquad (4\text{-}82)$$

$$M_q^* = M_q + M_{\dot{w}}U_0 \qquad (4\text{-}83)$$

$$M_\delta^* = M_\delta + M_{\dot{w}}Z_\delta \qquad (4\text{-}84)$$

we will have:

$$\dot{q} = (M_u^*)u + (M_w^*)w + (M_q^*)q + (M_\delta^*)\delta \qquad (4\text{-}85)$$

It is further desirable to include the height of the vehicle as another variable of interest. The equation for the height is obtained from the original Z-equation Equation 2-134 which for $\gamma_0 = 0$ becomes

$$\frac{1}{m} Z = \dot{w} - U_0 q \tag{4-86}$$

The left-hand side of this equation is just the z-axis acceleration, i.e.

$$-a_z = -\ddot{h} = \dot{w} - U_0 q \tag{4-87}$$

Integrating the last equation with respect to time we obtain

$$\dot{h} = -w + U_0 \theta \tag{4-88}$$

the constant of integration being the initial w-velocity perturbation which is zero. Equation 4-88 describes the height perturbation around the nominal value.

Collecting Equations 4-77, 4-78, 4-85 and 4-88 in matrix form we obtain the so called state variable representation of the longitudinal dynamics model, the state variables being, u,w,q,$\theta$, and h.

$$\begin{vmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \\ \dot{h} \end{vmatrix} = \begin{vmatrix} X_u & X_w & 0 & -g & 0 \\ Z_u & Z_w & U_0 & 0 & 0 \\ M^*_u & M^*_w & M^*_q & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & U_0 & 0 \end{vmatrix} \cdot \begin{vmatrix} u \\ w \\ q \\ \theta \\ h \end{vmatrix} + \begin{vmatrix} X_\delta \\ Z_\delta \\ M^*_\delta \\ 0 \\ 0 \end{vmatrix} \delta \tag{4-89}$$

State variable representation form is perfectly suitable for computer solution. Many ordinary differential equation solver software routines are available for solution. Most of these are provided with plotting capabilities of the state variables.

The solution that follows corresponds to the given example with initial conditions and control inputs indicated in the summary of specifications tables.

It can be seen from the plots that the $v$-velocity, the $\theta$-pitch angle and the h-height are heavily excited in the phugoid motion, while the w-velocity and the q-pitch rate are heavily excited in the short period motion.

For the phugoid approximation the following model may be used:

$$\begin{vmatrix} \dot{v} \\ \dot{\theta} \\ \dot{h} \end{vmatrix} = \begin{vmatrix} X_v & -g & 0 \\ -Z_v/U_0 & 0 & 0 \\ 0 & U_0 & 0 \end{vmatrix} \cdot \begin{vmatrix} v \\ \theta \\ h \end{vmatrix} + \begin{vmatrix} X_\delta \\ Z_\delta/U_0 \\ 0 \end{vmatrix} \delta \qquad (4\text{-}90)$$

while for the short period approximation:

$$\begin{vmatrix} \dot{w} \\ \dot{q} \end{vmatrix} = \begin{vmatrix} Z_v & U_0 \\ M_w^* & M_q^* \end{vmatrix} \cdot \begin{vmatrix} w \\ q \end{vmatrix} + \begin{vmatrix} Z_\delta \\ M_\delta^* \end{vmatrix} \delta \qquad (4\text{-}91)$$

## 4.5 LONGITUDINAL AERODYNAMIC TRANSFER FUNCTIONS

Up to now we went through the analysis of the dynamical behavior of the flight vehicle, by making use of the mathematical model that adequately characterizes the longitudinal characteristic motions. In any dynamical system

two types of variables exist, the input and the output variables. The input variables influence the output variables in a manner determined by the dynamics of the system.

The objective of this section is that of determining the nature of those input - output relations called transfer functions.

For the longitudinal model, the input variable was the elevator control deflection $\delta$ and perhaps the initial conditions or disturbances that can be treated as inputs, while the output variables were $\upsilon$,w,q,$\theta$ and h.

The corresponding transfer functions are obtained by solving simultaneously at a time, the transformed equations of motion for the output variable of interest, keeping all initial conditions zero.

The transfer function $\Theta(s)/\delta(s)$ is

$$\frac{\Theta(s)}{\delta(s)} = \frac{\begin{vmatrix} s-X_u & -X_w & 0 & X_\delta \\ -Z_u & s-Z_w & -U_0 & Z_\delta \\ -M_w & -sM_{\dot{w}}-M_w & s-M_q & M_\delta \\ 0 & 0 & 1 & 0 \end{vmatrix}}{As^4+Bs^3+Cs^2+Ds+E} = \frac{A_\theta s^2+B_\theta s+C_\theta}{As^4+Bs^3+Cs^2+Ds+E} \quad (4\text{-}92)$$

where

$$A_\theta = M_\delta + M_{\dot{w}} Z_\delta \quad (4\text{-}93)$$

$$B_\theta = X_\delta(Z_u M_{\dot{w}} + M_u) + Z_\delta(M_w - X_u M_{\dot{w}}) - M_\delta(X_u + Z_w) \quad (4\text{-}94)$$

$$C_\theta = X_\delta(Z_u M_w - Z_w M_u) + Z_\delta(M_u X_w - M_w X_u) + M_\delta(Z_w X_u - X_w Z_u) \quad (4\text{-}95)$$

or

$$\frac{\theta(s)}{\delta(s)} = \frac{A_\theta (s+z_{\theta_1})(s+z_{\theta_2})}{As^4+Bs^3+Cs^2+Ds+E} \qquad (4\text{-}96)$$

where $z_{\theta_1}$, $z_{\theta_2}$ are the roots of the numerator polynomial called zeroes of the transfer function.

Similarily we can derive

$$\frac{q(s)}{\delta(s)} = \frac{s\,\theta(s)}{\delta(s)} = \frac{A_\theta\, s\,(s+z_{\theta_1})(s+z_{\theta_2})}{As^4+Bs^3+Cs^2+Ds+E} \qquad (4\text{-}97)$$

$$\frac{w(s)}{\delta(s)} = \frac{A_w s^3+B_w s^2+C_w s+D_w}{As^4+Bs^3+Cs^2+Ds+E} \qquad (4\text{-}98)$$

where

$$A_w = Z_\delta \qquad (4\text{-}99)$$

$$B_w = X_\delta Z_u - Z_\delta (X_u + M_q) + M_\delta U_o \qquad (4\text{-}100)$$

$$C_w = X_\delta (U_o M_u - Z_u M_q) + Z_\delta X_u M_q - U_o M_\delta X_u \qquad (4\text{-}101)$$

$$D_w = g\,(Z_\delta M_u - M_\delta Z_u) \qquad (4\text{-}102)$$

This transfer function may take the following forms

$$\frac{w(s)}{\delta(s)} = \frac{A_w (s+z_{w_1})(s^2+2\zeta_w \omega_w s+\omega_w^2)}{As^4+Bs^3+Cs^2+Ds+E} \qquad (4\text{-}103)$$

or

$$\frac{w(s)}{\delta(s)} = \frac{A_w (s+z_{w_1})(s+z_{w_2})(s+z_{w_3})}{As^4+Bs^3+Cs^2+Ds+E} \qquad (4\text{-}104)$$

We can therefore have three zeroes, two complex and one real or three real.

Other longitudinal transfer functions are:

$$\frac{u(s)}{\delta(s)} = \frac{A_u s^3+B_u s^2+C_u s+D_u}{As^4+Bs^3+Cs^2+Ds+E} \qquad (4\text{-}105)$$

where

$$A_u = X_\delta \qquad (4\text{-}106)$$

$$B_u = -X_\delta (Z_w + M_q + U_0 M_{\dot{w}}) + Z_\delta X_w \qquad (4\text{-}107)$$

$$C_u = X_\delta (Z_w M_q - U_0 M_w) - Z_\delta (X_w M_q + g M_{\dot{w}}) + M_\delta (U_0 X_w - g) \qquad (4\text{-}108)$$

$$D_u = g (M_\delta Z_w - Z_\delta M_w) \qquad (4\text{-}109)$$

This transfer function may also take the forms

$$\frac{u(s)}{\delta(s)} = \frac{A_u (s + z_{u_1})(s^2 + 2\zeta_u \omega_u s + \omega_u^2)}{As^4 + Bs^3 + Cs^2 + Ds + E} \qquad (4\text{-}110)$$

or

$$\frac{u(s)}{\delta(s)} = \frac{A_u (s + z_{u_1})(s + z_{u_2})(s + z_{u_3})}{As^4 + Bs^3 + Cs^2 + Ds + E} \qquad (4\text{-}111)$$

Similarily

$$\frac{h(s)}{\delta(s)} = \frac{A_h s^3 + B_h s^2 + C_h s + D_h}{s(As^4 + Bs^3 + Cs^2 + Ds + E)} \qquad (4\text{-}112)$$

where

$$A_h = -Z_\delta \qquad (4\text{-}113)$$

$$B_h = -X_\delta Z_u + Z_\delta (M_q + U_0 M_{\dot{w}} + X_u) \qquad (4\text{-}114)$$

$$C_h = X_\delta Z_u (M_q + U_0 M_{\dot{w}}) - Z_\delta [X_u (M_q + U_0 M_{\dot{w}}) - U_0 M_w] - M_\delta U_0 M_w \qquad (4\text{-}115)$$

$$\begin{aligned} D_h = {} & -X_\delta U_0 (Z_w M_u - M_w Z_u) + Z_\delta [M_u (U_0 X_w - g) - U_0 M_w X_u] \\ & + M_\delta [U_0 Z_w X_u - Z_u (U_0 X_w - g)] \end{aligned} \qquad (4\text{-}116)$$

and may take for forms

$$\frac{h(s)}{\delta(s)} = \frac{A_h (s + z_{h_1})(s^2 + 2\zeta_h \omega_h s + \omega_h^2)}{s(As^4 + Bs^3 + Cs^2 + Ds + E)} \qquad (4\text{-}117)$$

or

$$\frac{h(s)}{\delta(s)} = \frac{A_h (s + z_{h_1})(s + z_{h_2})(s + z_{h_3})}{s(As^4 + Bs^3 + Cs^2 + Ds + E)} \qquad (4\text{-}118)$$

Another transfer function of interest is that relating the z-axis acceleration to the control input, i.e.

$$\frac{a_z(s)}{\delta(s)} = \frac{-s^2 h(s)}{\delta(s)} = \frac{A_h\, s\,(s+z_{h_1})\,(s^2+2\zeta_h\omega_h s+\omega_h^2)}{As^4+Bs^3+Cs^2+Ds+E} \qquad (4\text{-}119)$$

or

$$\frac{a_z(s)}{\delta(s)} = \frac{A_h\, s\,(s+z_{h_1})(s+z_{h_2})(s+z_{h_3})}{As^4+Bs^3+Cs^2+Ds+E} \qquad (4\text{-}120)$$

Those transfer functions are used in automatic or manual flight control systems, since they specify the input - output relationships of the variables. For example, to find the open loop response $\theta(t)$ due to a unit step elevator input we use the transfer function Equation 4-96

$$\theta(s) = \left[\frac{\theta(s)}{\delta(s)}\right]\delta(s) = \frac{A_\theta\, s\,(s+z_{\theta_1})(s+z_{\theta_2})}{As^4+Bs^3+Cs^2+Ds+E}\cdot\delta(s) \qquad (4\text{-}121)$$

Taking the inverse Laplace Transform of Equation 4-121 we obtain the time response.

## 4.6 LONGITUDINAL EQUATIONS IN NON-DIMENSIONAL SYSTEMS

The equations of motion of a flight vehicle, can be written in a number of different forms, depending on the particular axes system selected for the definition of variables and the stability analysis.

We have just described the body axes system in the dimensional form, in which most stability work is carried out [Ref. 1].

In most aerodynamic problems it is more convenient to use non-dimensional coefficients to represent the parameters involved. By this non-dimensional system, the equations of motion can be described in the body axes or wind axes system as well.

The non-dimensional coefficients have the advantage that the effect of speed, size and air density are automatically accounted. However, it is required to express the mass and the time in non-dimensional forms as well. Therefore, one cannot obtain real time solutions.

In the non-dimensional body axes system the longitudinal equations of motion are given as reference [Ref. 2].

$$\begin{vmatrix} 2\mu s - C_{x_u} & -C_{x_\alpha} & C_{L_0} \\ 2C_{L_0} - C_{z_u} & 2\mu s - sC_{z_{\dot\alpha}} - C_{z_\alpha} & -2\mu s - sC_{z_q} \\ -C_{m_u} & -sC_{m_{\dot\alpha}} - C_{m_\alpha} & \iota_B s^2 - sC_{m_q} \end{vmatrix} \cdot \begin{vmatrix} u \\ \alpha \\ \theta \end{vmatrix} = \begin{vmatrix} 0 \\ C_{z_n} \\ sC_{m_{\dot n}} + C_{m_n} \end{vmatrix} \eta \qquad (4\text{-}122)$$

where the non-dimensional mass, inertia and time are expressed as:

$$\mu = m/\rho S \ell \qquad (4\text{-}123)$$

$\ell$ being the characteristic length taken as the half of the aerodynamic chord

$$\iota_B = \frac{I_{yy}}{\rho S \ell^3} \qquad (4\text{-}124)$$

$$t^* = \ell/U_0 \qquad (4\text{-}125)$$

In the non-dimensional wind axes system the longitudinal equations are [Ref. 3]

$$\begin{vmatrix} s + C_D & \frac{1}{2}(C_{D_\alpha} - C_L) & \frac{1}{2}C_L \\ C_L & s + \frac{1}{2}C_{L_\alpha} & -s \\ 0 & sC_{m_{\dot{\alpha}}} + C_{m_\alpha} & sC_{m_{\dot{\theta}}} - s^2 I_{y_1} \end{vmatrix} \begin{vmatrix} v \\ \alpha \\ \theta \end{vmatrix} = \begin{vmatrix} 0 \\ C_{Z_\eta} \\ sC_{m_{\dot{\eta}}} + C_{m_\eta} \end{vmatrix} \eta \qquad (4\text{-}126)$$

where the non-dimensional mass, inertia and time are expressed as:

$$\mu = m/\rho S \bar{c} \qquad (4\text{-}127)$$

$$I_{y_1} = 2\,\frac{I_{yy}/m}{\mu \bar{c}^2} \qquad (4\text{-}128)$$

$$\tau = m/\rho S U_0 \qquad (4\text{-}129)$$

$$t^* = t/\tau \qquad (4\text{-}130)$$

## TABLE 4-1

## SPECIFICATIONS FOR COMPUTER SOLUTION OF THE LONGITUDINAL EQUATIONS DEDICATED FOR PHUGOID MOTION DUE TO INITIAL CONDITIONS

A. Variables and Initial Conditions

$u = 5.0$ ft/sec
$w = 2.5$ ft/sec
$q = 0.05$ rad/sec
$\theta = 0.075$ rad
$h = 10.0$ ft
$t = 0.0$

B. Stability Derivatives and Constants

$X_u = -0.0097$ $Z_u = -0.0955$ $M_u = 0.0$ $g = 32.174$
$X_w = 0.0016$ $Z_w = -1.43$ $M_w = -0.0235$ $U_0 = 660.0$
$X_\delta = 0.0$ $Z_\delta = -69.8$ $M_{\dot{w}} = -0.0013$
$M_q = -1.92$
$M_\delta = -26.10$

C. Special Functions

$$M_u^* = M_u + M_{\dot{w}} Z_u$$

$$M_w^* = M_w + M_{\dot{w}} Z_w$$

$$M_q^* = M_q + M_{\dot{w}} U_0$$

$$M_\delta^* = M_\delta + M_{\dot{w}} Z_\delta$$

D. Control Input

$\delta = 0.0$

E. Derivatives

$$\dot{u} = X_u u + X_w w - g\theta + X_\delta \delta$$

$$\dot{w} = Z_u u + Z_w w + U_0 q + Z_\delta \delta$$

$$\dot{q} = M_u^* u + M_w^* w + M_q^* q + M_\delta^* \delta$$

$$\dot{\theta} = q$$

$$\dot{h} = -w + U_0 \theta$$

F. Outputs

$u$, $w$, $q$, $\theta$, $h$ vs time at interval 0.0625
end calculation when time > 250.0

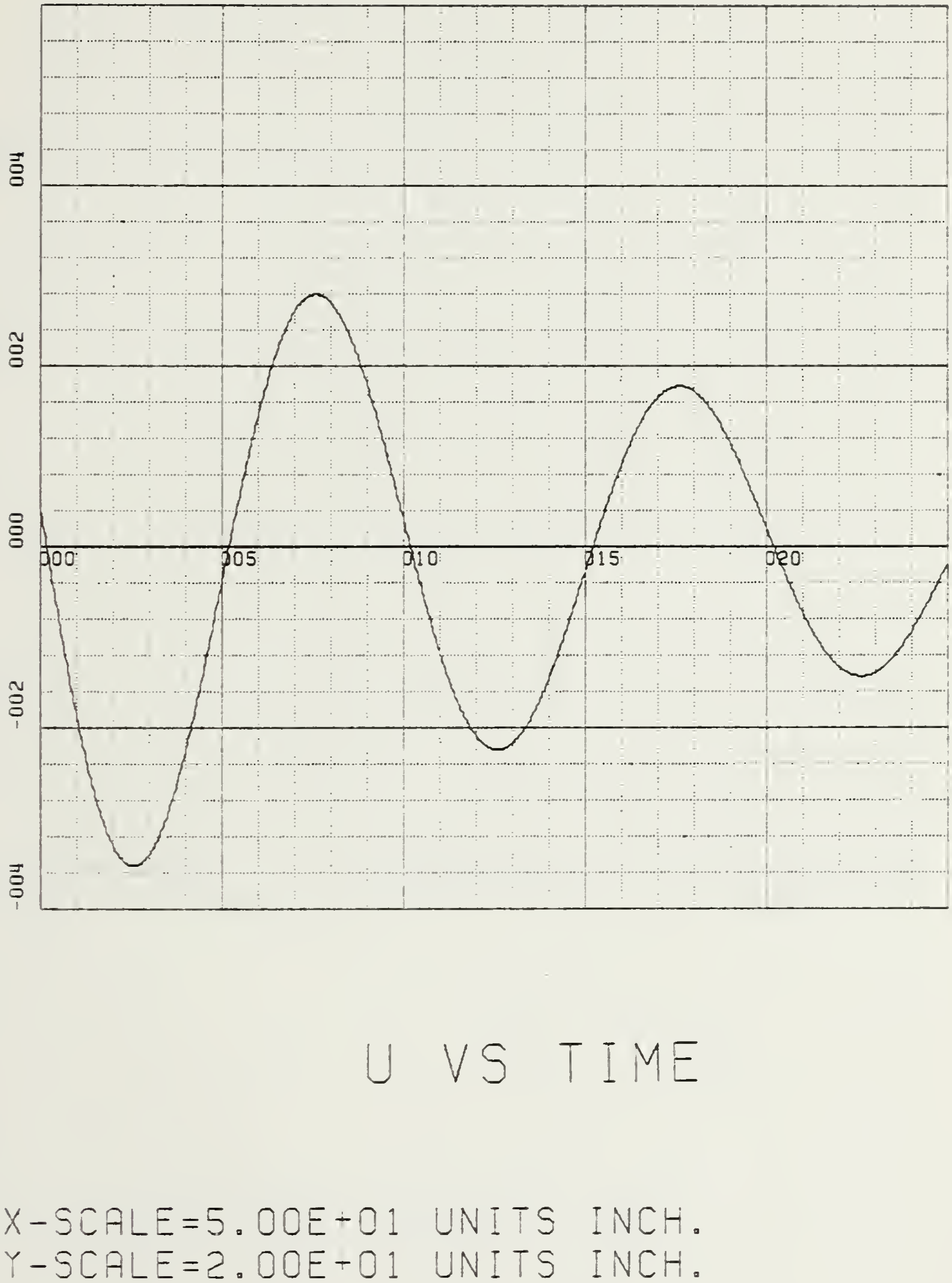
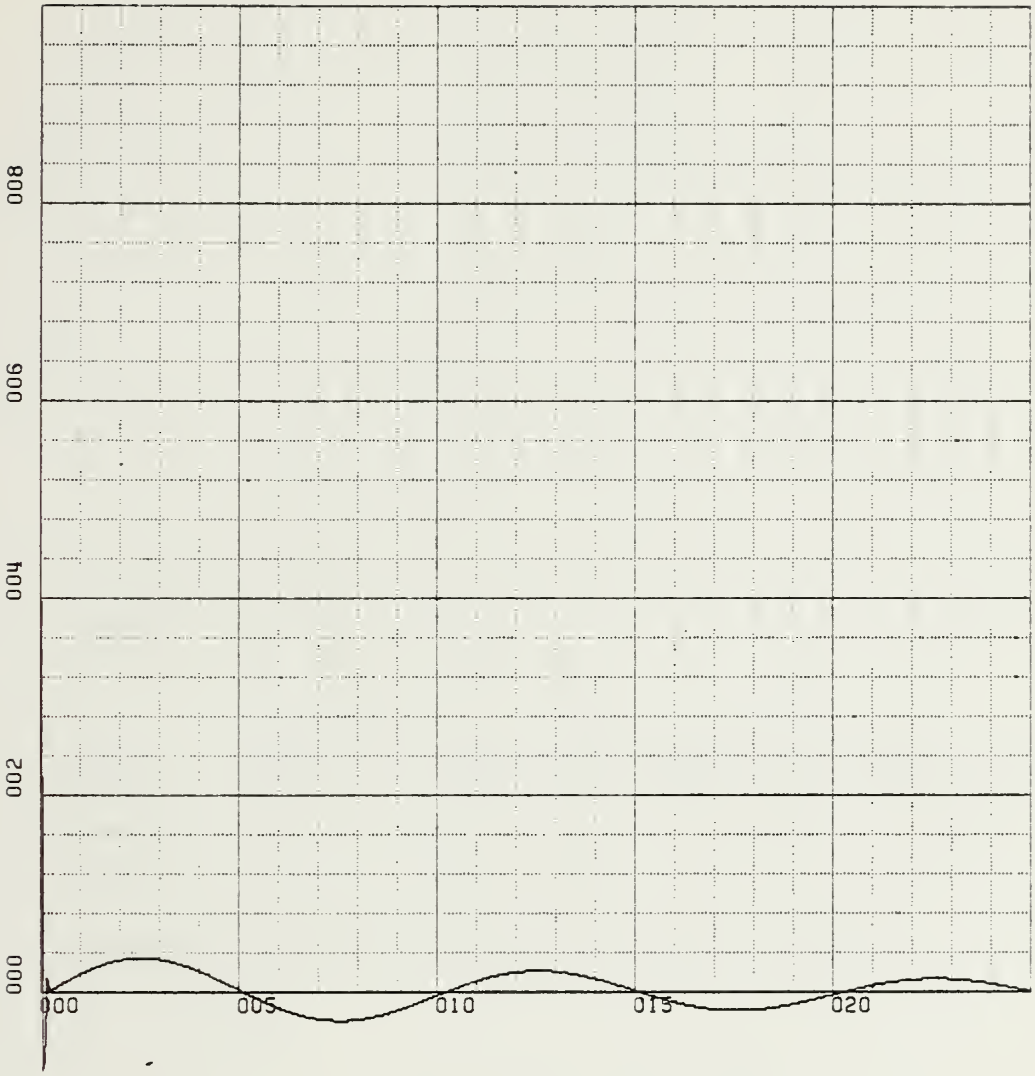


Figure 4-5

$v$-Velocity vs Time for Specifications of Table 4-1



W VS TIME

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

Figure 4-6

w-Velocity vs Time for Specifications of Table 4-1

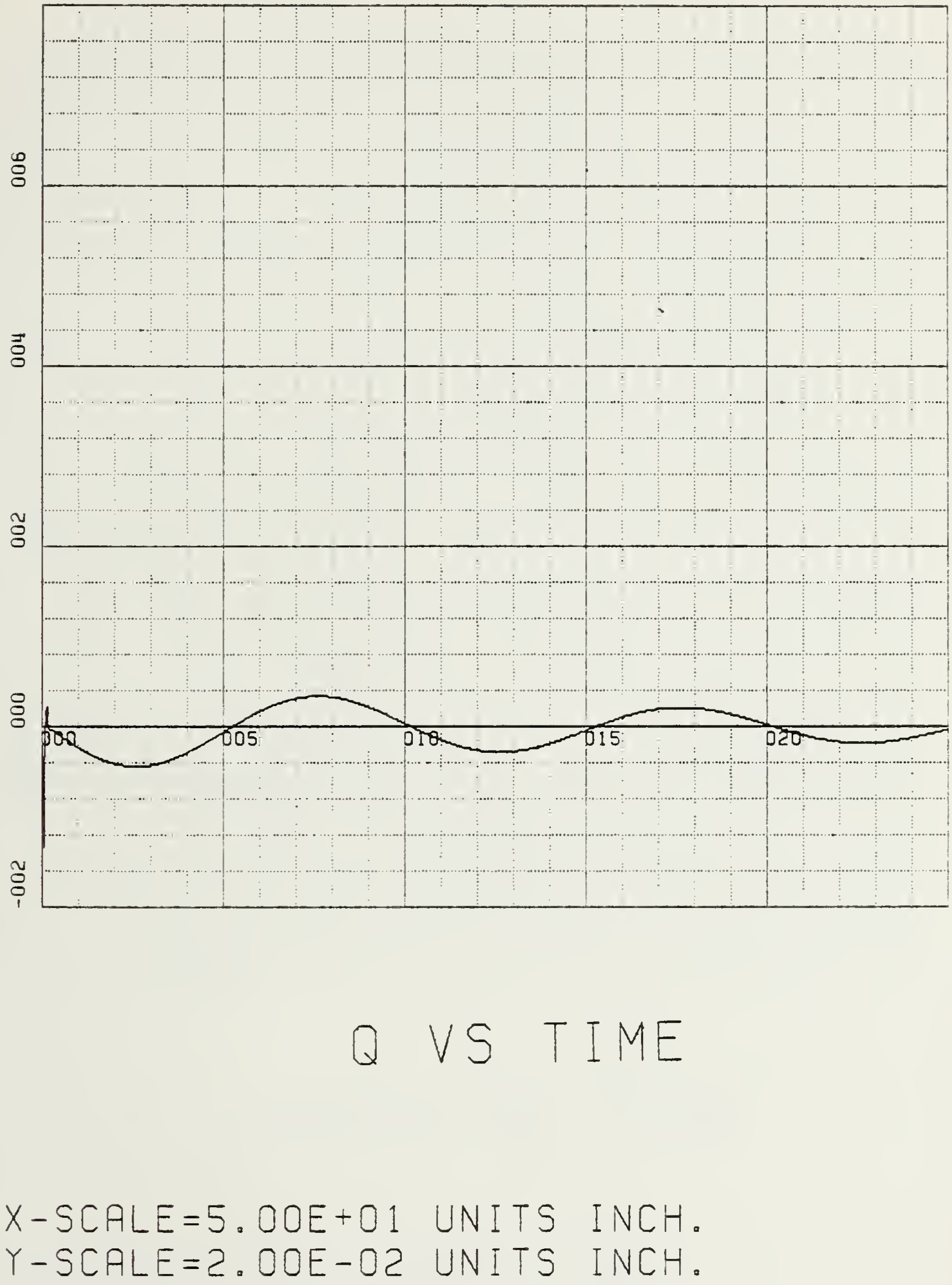
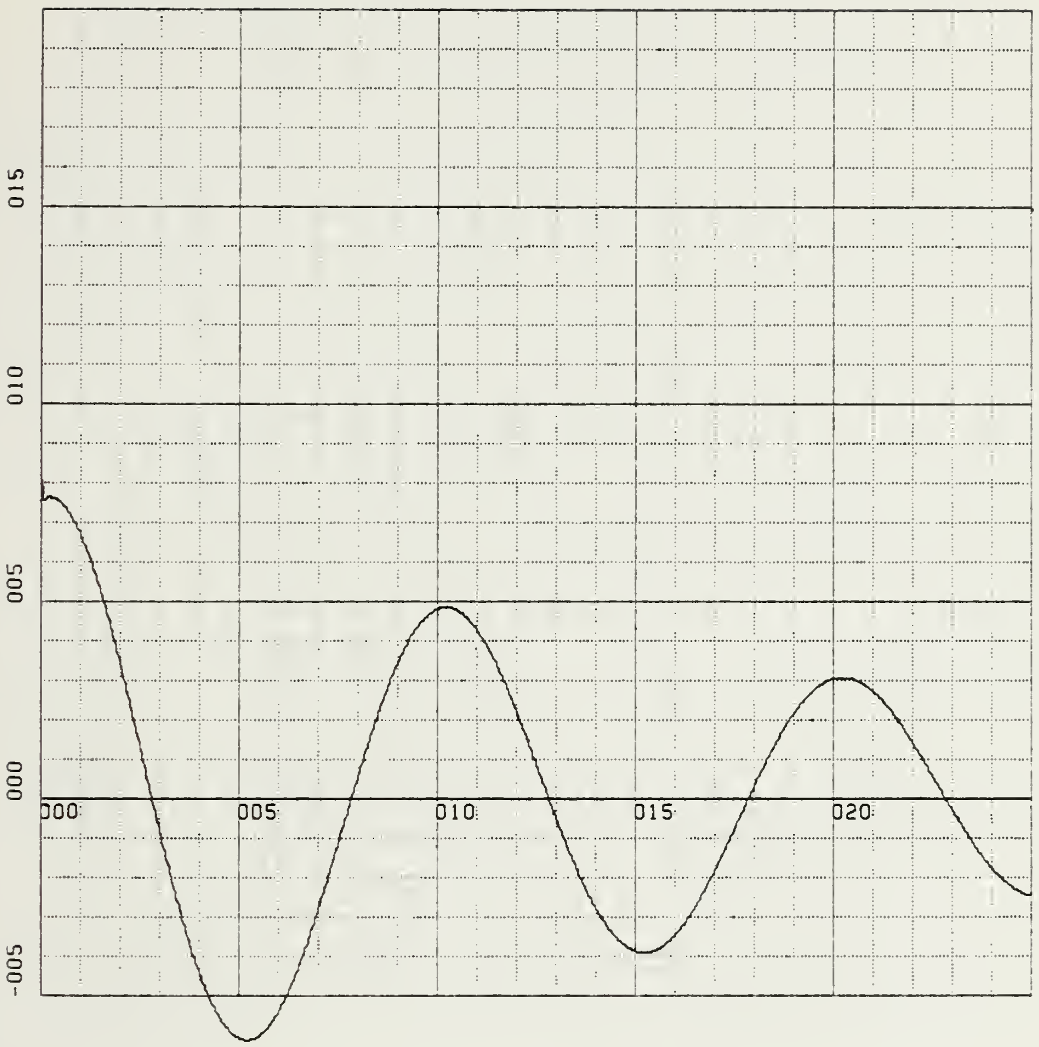


Figure 4-7

q-Pitch Rate vs Time for Specifications of Table 4-1



THETA VS TIME

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-02 UNITS INCH.

Figure 4-8

θ-Pitch Angle vs Time for Specifications of Table 4-1

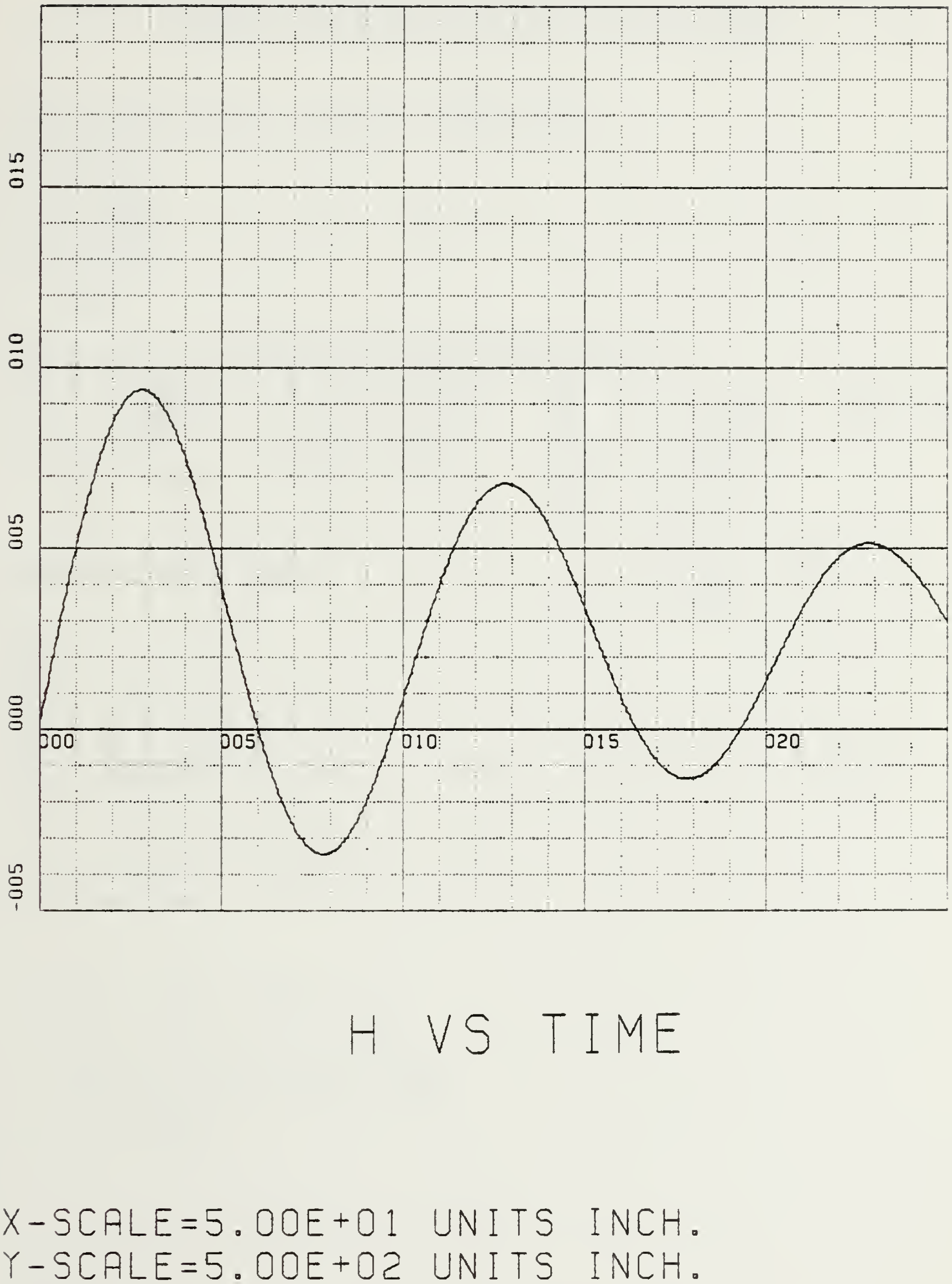


Figure 4-9

h-Height vs Time for Specifications of Table 4-1

## TABLE 4-2

## SPECIFICATIONS FOR COMPUTER SOLUTION OF THE LONGITUDINAL EQUATIONS DEDICATED FOR SHORT PERIOD MOTION DUE TO INITIAL CONDITIONS

A. Variables and Initial Conditions

$\upsilon = 5.0$
$w = 2.5$
$q = 0.05$
$\theta = 0.075$
$h = 10.0$
$t = 0.0$

B. Stability Derivatives and Constants

$X_\upsilon = -0.0097$
$X_w = 0.0016$
$X_\delta = 0.0$

$Z_\upsilon = -0.0955$
$Z_w = -1.43$
$Z_\delta = -69.8$

$M_\upsilon = 0.0$
$M_w = -0.02355$
$M_{\dot{w}} = -0.0013$
$M_q = -1.92$
$M_\delta = -26.10$

$g = 32.174$
$\upsilon_0 = 660.0$

C. Special Functions

$$M_\upsilon^* = M_\upsilon + M_{\dot{w}} Z_\upsilon$$
$$M_w^* = M_w + M_{\dot{w}} \cdot Z_w$$
$$M_q^* = M_q + M_{\dot{w}} U_0$$
$$M_\delta^* = M_\delta + M_{\dot{w}} Z_\delta$$

D. Control Input

$\delta = 0.0$

E. Derivatives

$$\dot{\upsilon} = X_\upsilon \upsilon + X_w w - g\theta + X_\delta \delta$$
$$\dot{w} = Z_\upsilon \upsilon + Z_w w + U_0 q + Z_\delta \delta$$
$$\dot{q} = M_\upsilon^* \upsilon + M_w^* w + M_q^* q + M_\delta^* \delta$$
$$\dot{h} = -w + U_0 \theta$$
$$\dot{\theta} = q$$

F. Outputs

$\upsilon$, $w$, $q$, $\Theta$, $h$ vs time at interval 0.00125
end calculation when time > 5.0

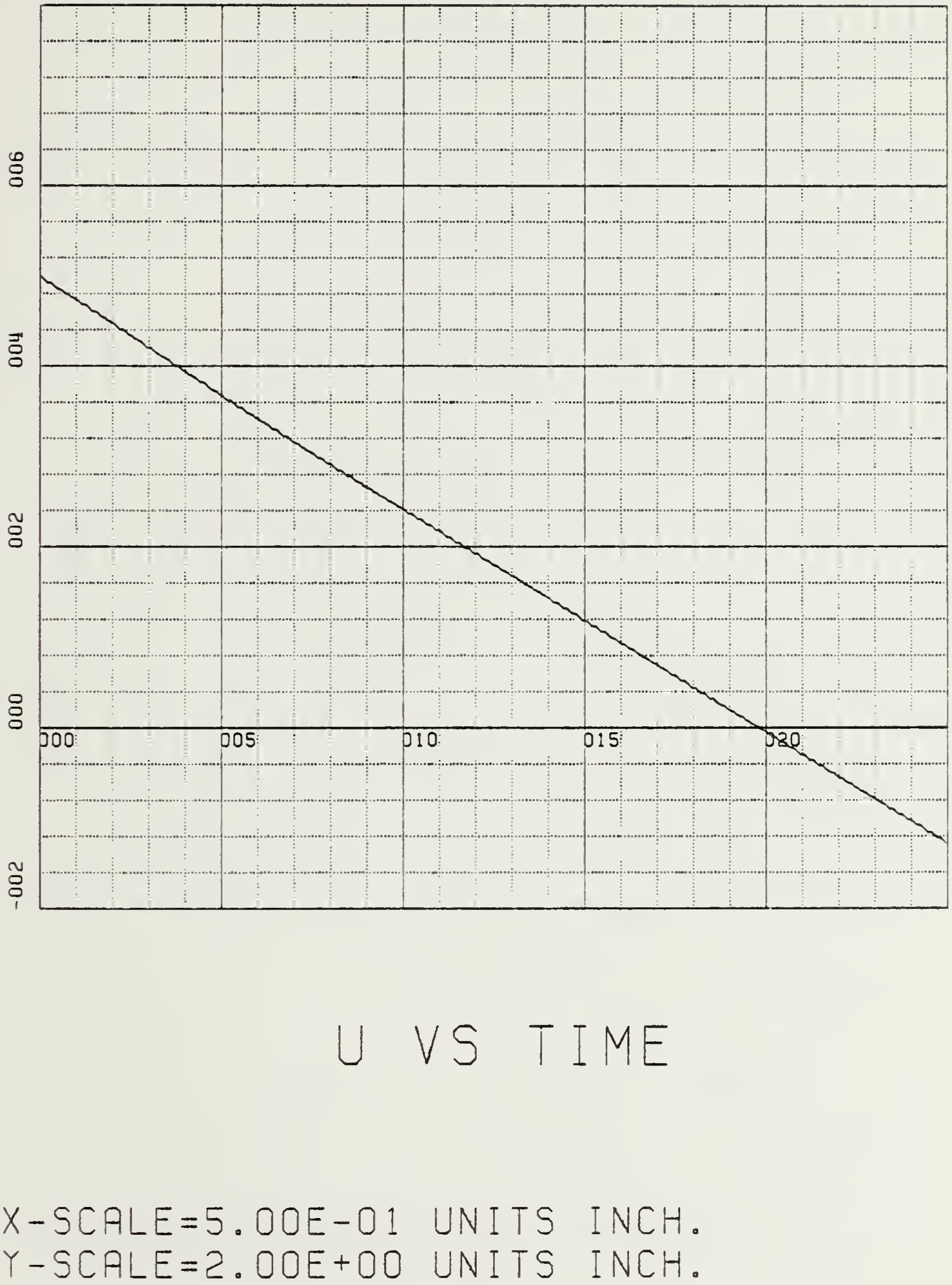


Figure 4-10

$\upsilon$-Velocity vs Time for Specifications of Table 4-2

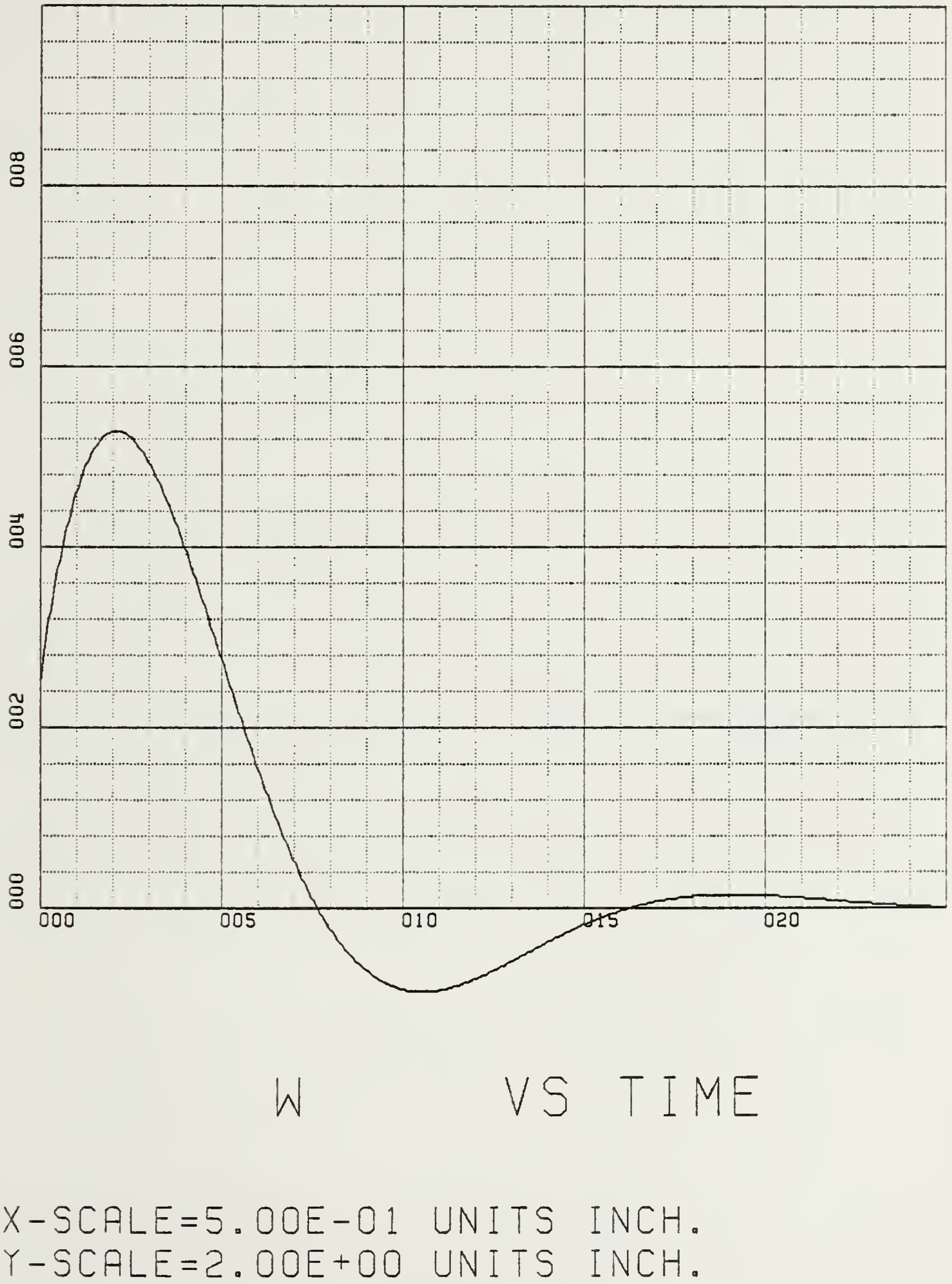
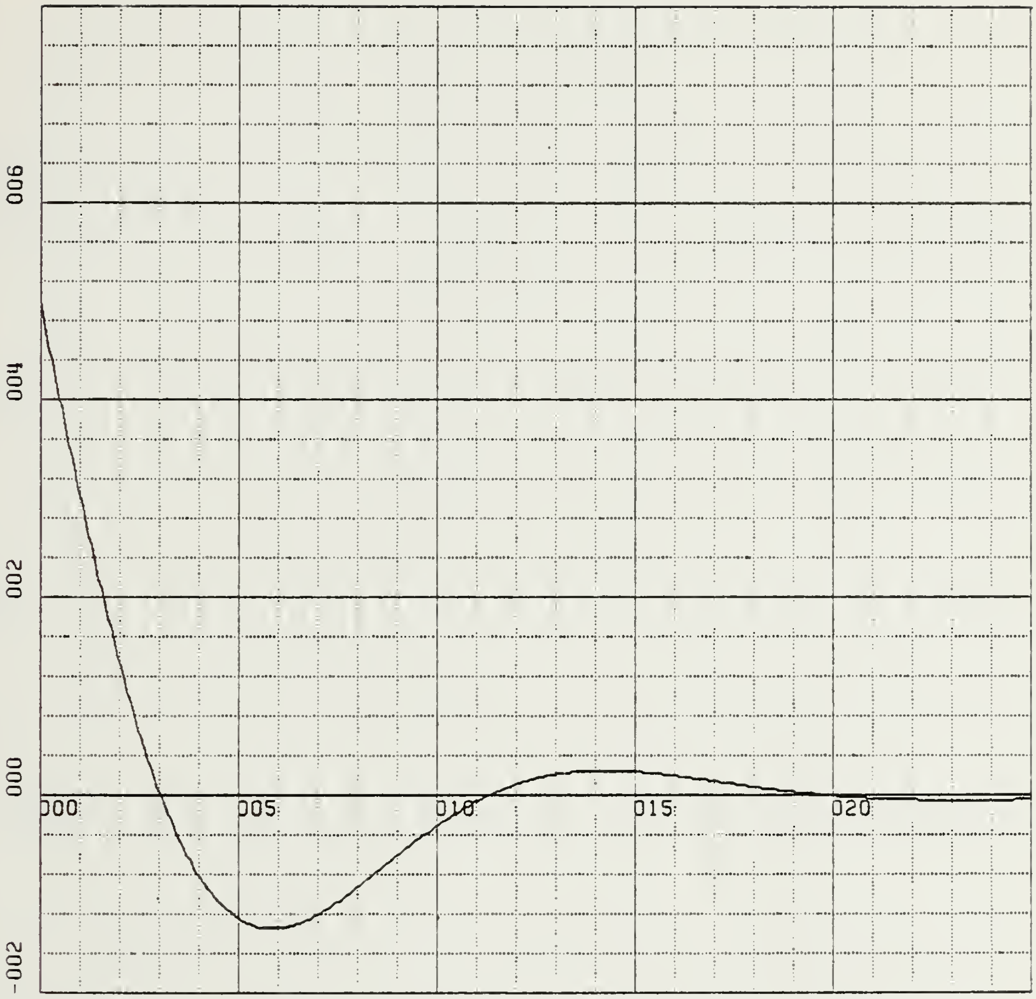


Figure 4-11

w-Velocity vs Time for Specifications of Table 4-2

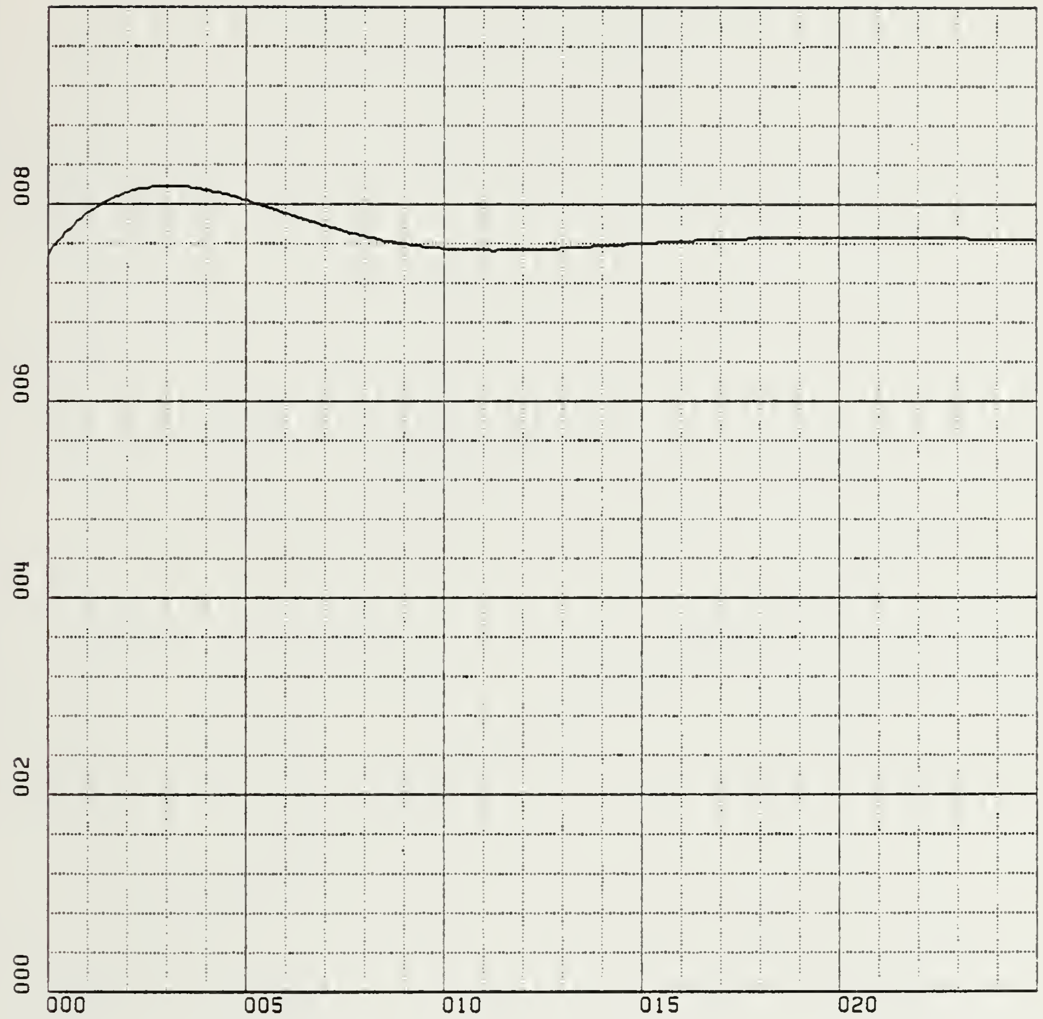


Q VS TIME

X-SCALE=5.00E-01 UNITS INCH.
Y-SCALE=2.00E-02 UNITS INCH.

Figure 4-12

q-Pitch Rate vs Time for Specifications of Table 4-2



THETA VS TIME

X-SCALE=5.00E-01 UNITS INCH.
Y-SCALE=2.00E-02 UNITS INCH.

Figure 4-13

θ-Pitch Angle vs Time for Specifications of Table 4-2

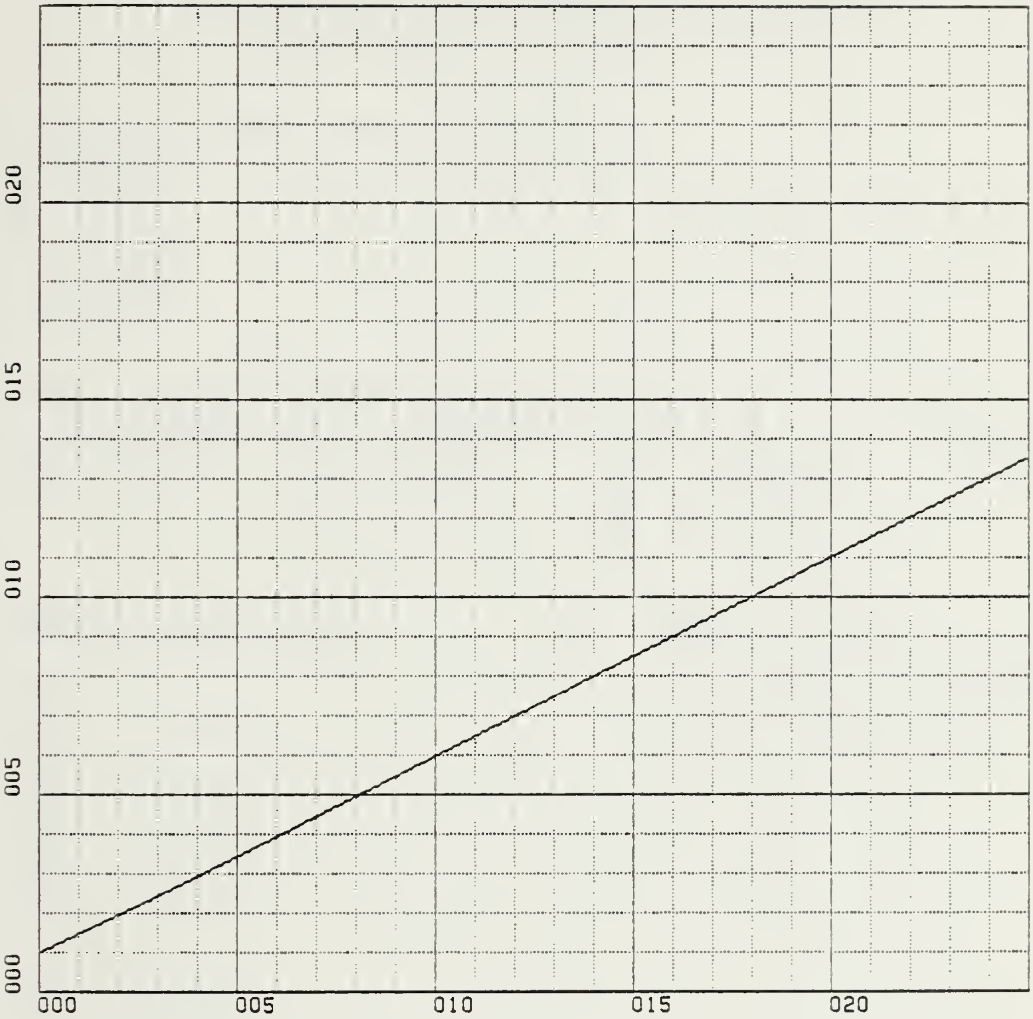


Figure 4-14

h-Height vs Time for Specifications of Table 4-2

## TABLE 4-3

## SPECIFICATIONS FOR COMPUTER SOLUTION OF THE LONGITUDINAL EQUATIONS DUE TO INITIAL CONDITIONS AND CONTROL INPUTS

A. Variables and Initial Conditions

$\upsilon = 5.0$
$w = 2.5$
$q = 0.05$
$\theta = 0.075$
$h = 10.0$
$t = 0.0$

B. Stability Derivatives and Constants

$X_\upsilon = -0.0097$
$X_w = 0.0016$
$X_\delta = 0.0$

$Z_\upsilon = -0.0955$
$Z_w = -1.43$
$Z_\delta = -69.8$

$M_\upsilon = 0.0$
$M_w = -0.0235$
$M_{\dot{w}} = -0.0013$
$M_q = -1.92$
$M_\delta = -26.10$

$g = 32.174$
$U_0 = 660.0$

C. Special Functions

$$M_\upsilon^* = M_\upsilon + M_{\dot{w}} Z_\upsilon$$

$$M_w^* = M_w + M_{\dot{w}} Z_w$$

$$M_q^* = M_q + M_{\dot{w}} U_0$$

$$M_\delta^* = M_\delta + M_{\dot{w}} Z_\delta$$

D. Control Input

$\delta = 0.01$ for $0<t<10.0$
$\delta = -0.01$ for $10.0<t<20.0$
$\delta = 0.0$ for $t>20.0$

E. Derivatives

$$\dot{\upsilon} = X_\upsilon \upsilon + X_w w - g\theta + X_\delta \delta$$

$$\dot{w} = Z_\upsilon \upsilon + Z_w w + U_0 q + Z_\delta \delta$$

$$\dot{q} = M_\upsilon^* \upsilon + M_w^* w + M_q^* q + M_\delta^* \delta$$

$$\dot{h} = \dot{w} + U_0 \theta$$

$$\dot{\theta} = q$$

F. Outputs

$\upsilon$, $w$, $q$, $\theta$, $h$ vs time at interval 0.0625
end calculation when time $>$ 250.0

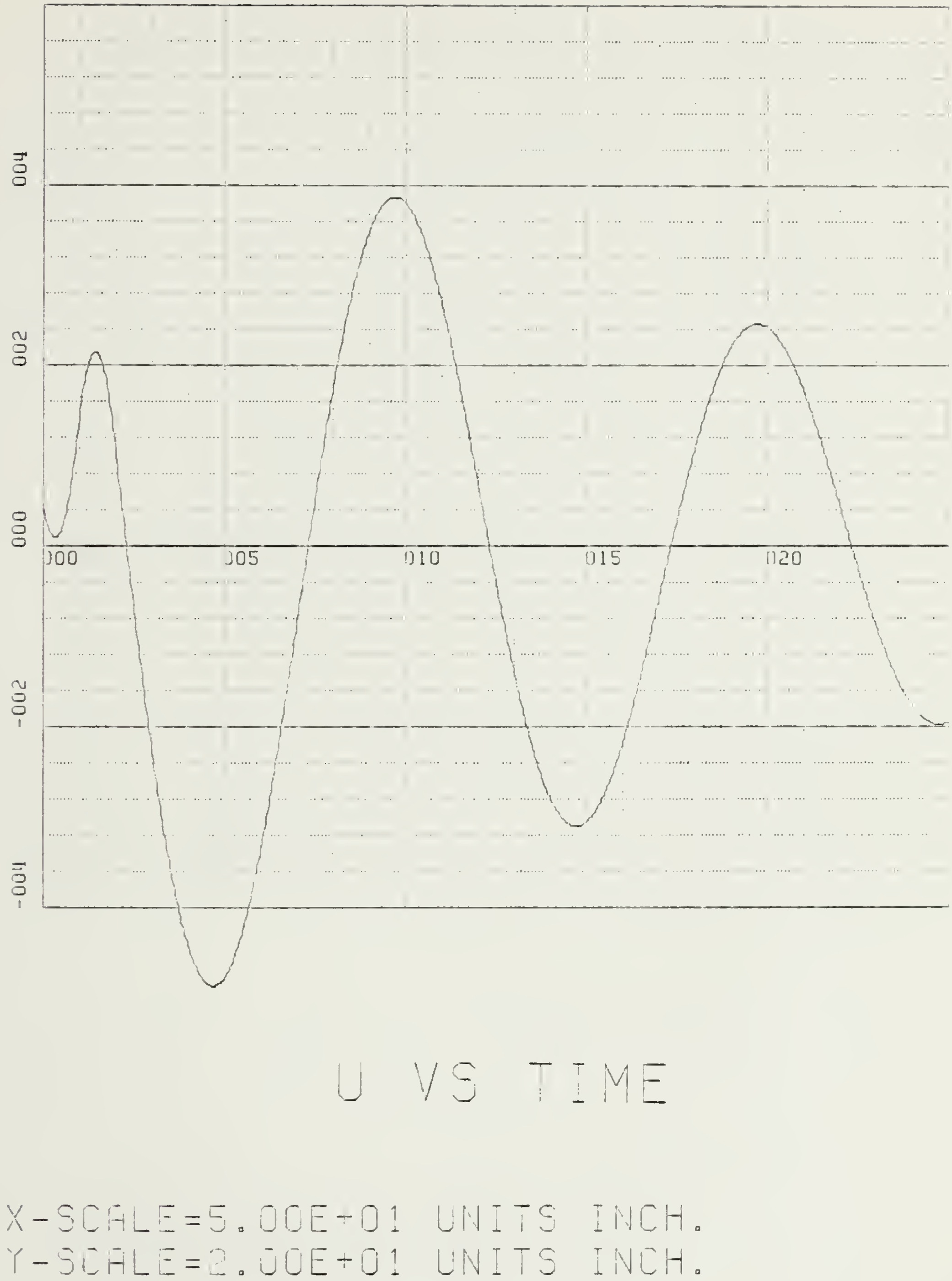


Figure 4-15

$\upsilon$-Velocity vs Time for Specifications of Table 4-3

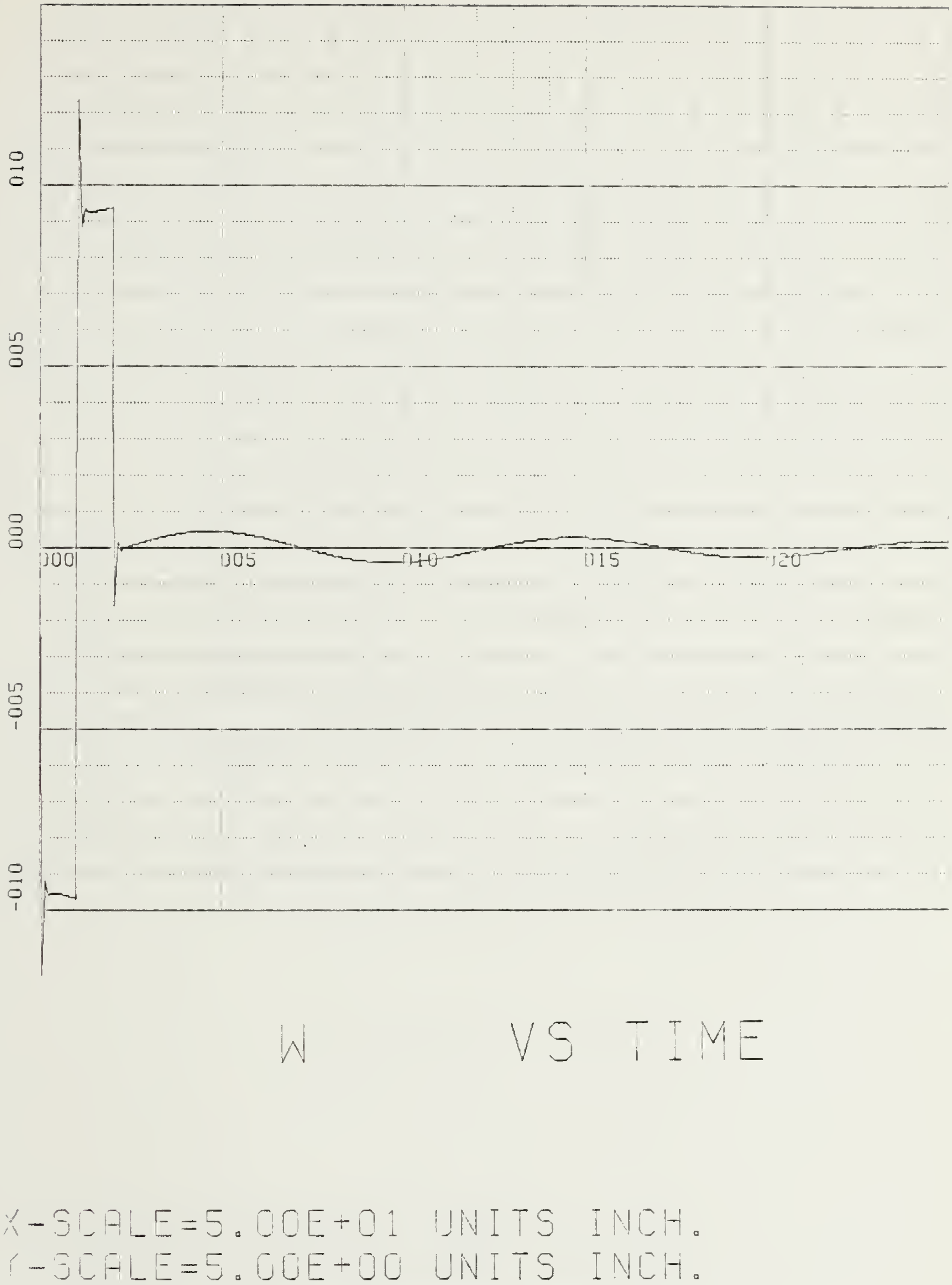


Figure 4-16

w-Velocity vs Time for Specifications of Table 4-3

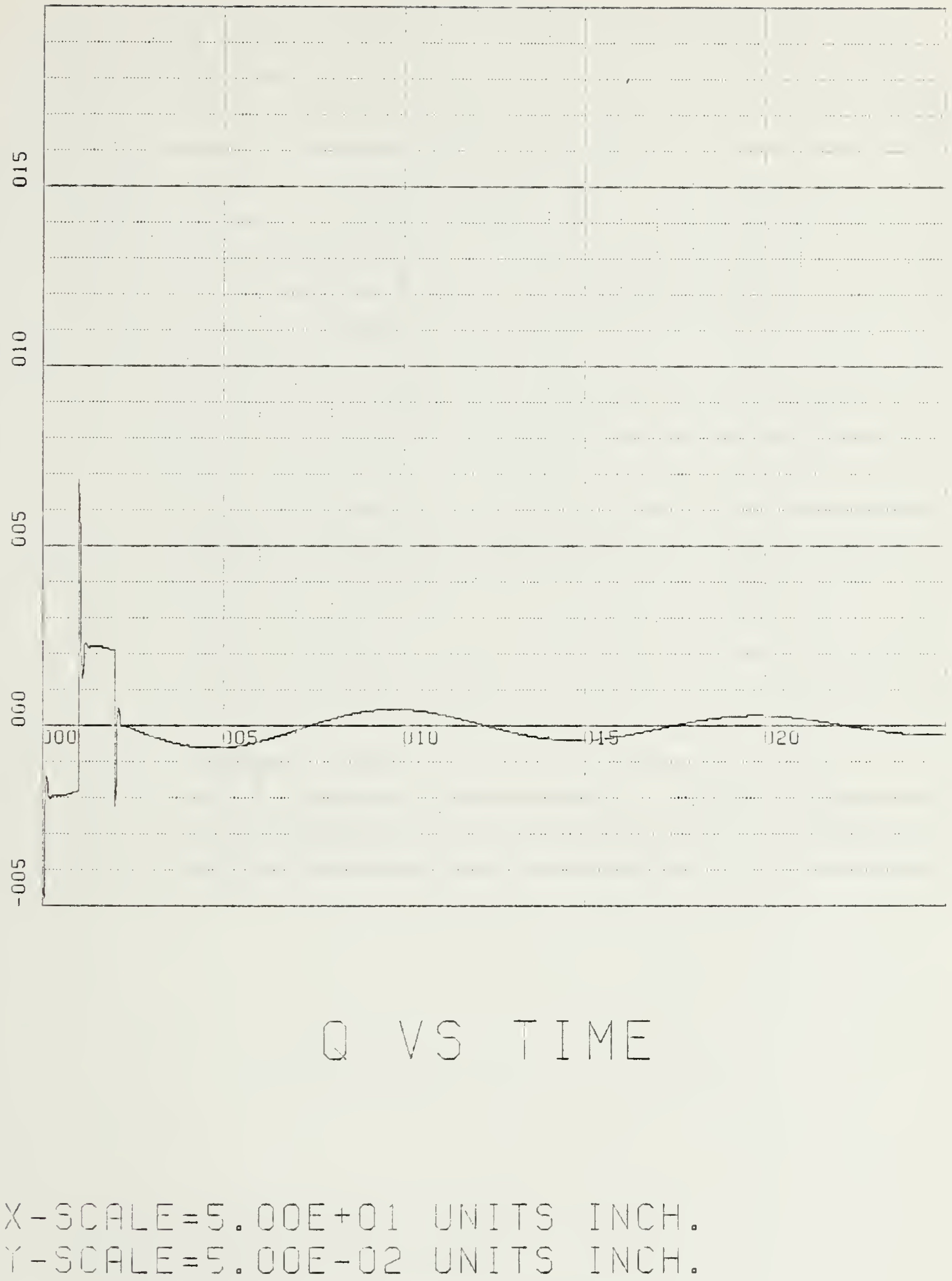


Figure 4-17

q-Pitch Rate vs Time for Specifications of Table 4-3

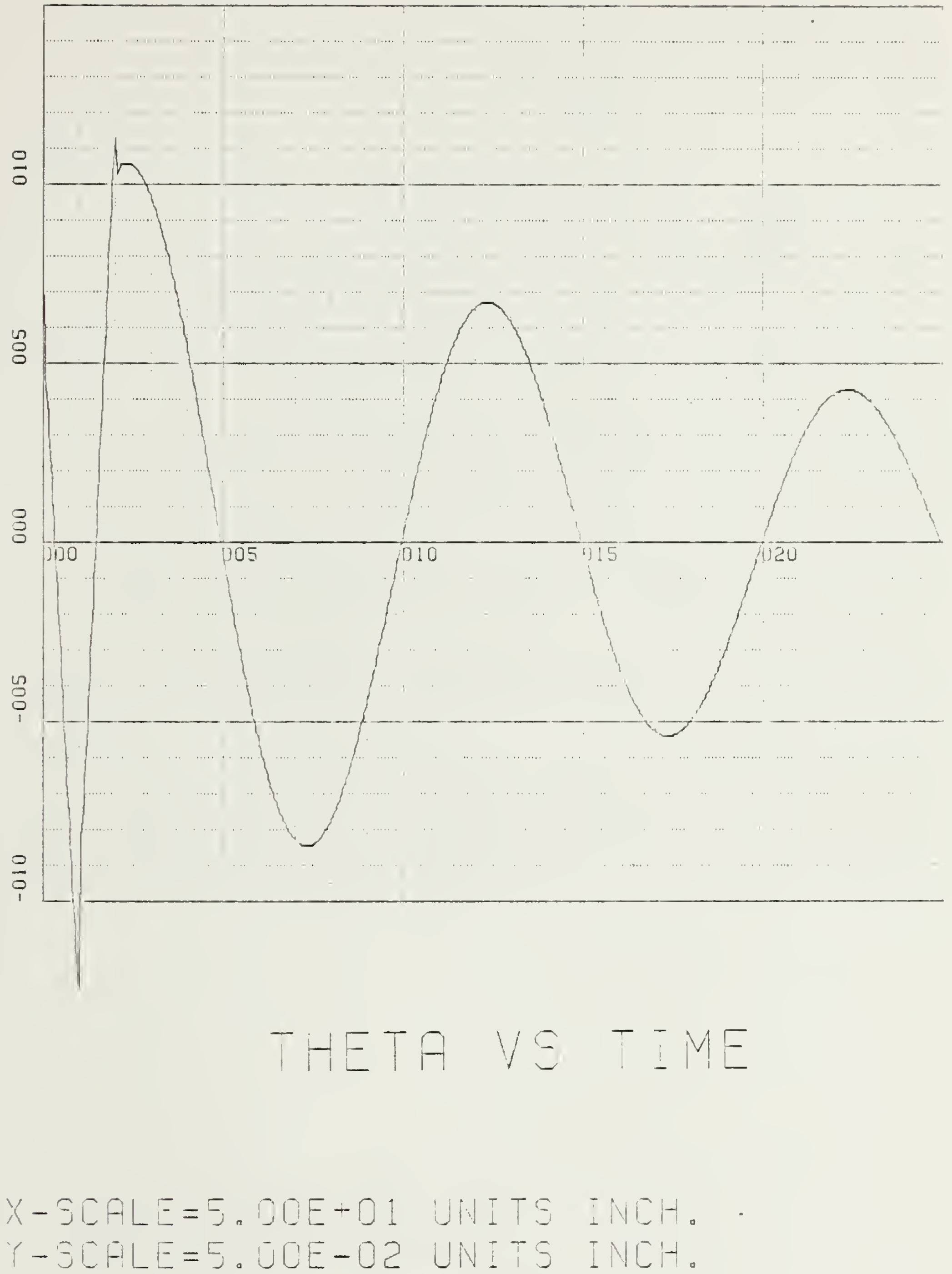


Figure 4-18

Θ -Pitch Angle vs Time for Specifications of Table 4-3

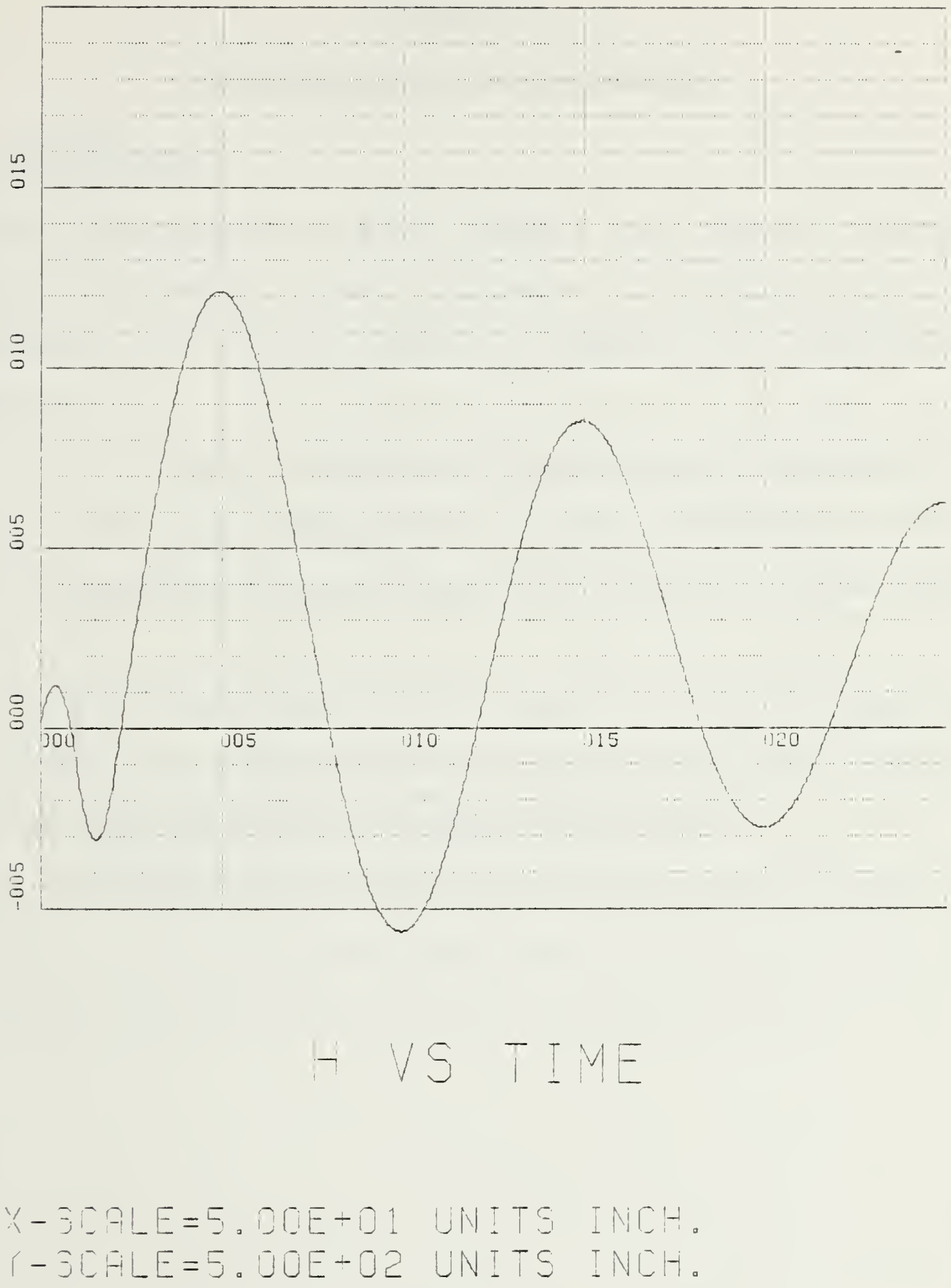


Figure 4-19

h-Height vs Time for Specifications of Table 4-3

# CHAPTER 5

# LATERAL DIRECTIONAL DYNAMICS

## 5.1 INTRODUCTION

In this chapter we will analyze the lateral directional motion of a conventional airplance, by folliwng the same treatment as in the preceding chapter on longitudinal dynamics.

In the lateral directional case also, any particular response will be associated as in the longitudinal case, by its corresponding characteristic root location in the complex plane.

Further simplified set of equations will be developed that apply to the individual modes of motion. Finally, computer solutions will be emphasized as well.

As an example, the lateral-directional motion of the same aircraft as in the preceding chapter will be analyzed with the lateral-directional parameters indicated below:

| | |
|---|---|
| altitude (ft) | 20.000 |
| weight (lb) | 30.500 |
| mach number | 0.638 |
| air speed (ft/sec) | 660 |

$Y_v = -0.0829$

$Y_\zeta = 7.656$

$L_\beta = -4.77$

$L_p = -1.695$

$L_r = 0.1776$

$L_\xi = 27.25$

$L_\zeta = 0.666$

$N_\beta = 3.55$

$N_p = -0.0025$

$N_r = -0.0957$

$N_\xi = -0.615$

$N_\zeta = -1.383$

$I_{xz}/I_{xx} = 0.0663$

$I_{xz}/I_{zz} = 0.0370$

## 5.2 CHARACTERISTIC POLYNOMIAL OF LATERAL-DIRECTIONAL MOTION

Lateral-directional equations of motion are further simplified by neglecting the effect of $Y_{\dot v}$, $Y_p$, $Y_r$, $L_{\dot v}$, and $N_{\dot v}$ derivatives which are of no importance in lateral directional dynamics. Furthermore, a straight level flight is assumed as the reference flight condition (i.e. $\gamma_0 = 0$). Hence, Equation 2-153 is reduced to:

$$\begin{vmatrix} s-Y_v & 0 & 1 & -g/U_0 & 0 \\ -L_\beta & s-L_p & -s\frac{I_{xz}}{I_{xx}}-L_r & 0 & 0 \\ -N_\beta & -s\frac{I_{xz}}{I_{zz}}-N_p & s-N_r & 0 & 0 \\ 0 & 1 & 0 & -s & 0 \\ 0 & 0 & 1 & 0 & -s \end{vmatrix} \begin{vmatrix} \beta \\ p \\ r \\ \phi \\ \psi \end{vmatrix} = \begin{vmatrix} Y_\delta^* \\ L_\delta \\ N_\delta \\ 0 \\ 0 \end{vmatrix} \delta \qquad (5\text{-}1)$$

The characteristic polynomial, as known, is found by evaluation of the above determinant which comes out to be:

$$As^4 + Bs^3 + Cs^2 + Ds + E = 0 \tag{5-2}$$

where

$$A = 1 - \frac{I_{xz}^2}{I_{xx} I_{zz}} \tag{5-3}$$

$$B = -Y_v \left(1 - \frac{I_{xz}^2}{I_{xx} I_{zz}}\right) - L_p - N_r - \frac{I_{xz}}{I_{xx}} N_p - \frac{I_{xz}}{I_{zz}} L_r \tag{5-4}$$

$$C = N_\beta + L_p(Y_v + N_r) + N_p\left(\frac{I_{xz}}{I_{xx}} Y_v - L_r\right) + Y_v\left(\frac{I_{xz}}{I_{zz}} L_r + N_r\right) + \frac{I_{xz}}{I_{zz}} L_\beta \tag{5-5}$$

$$D = -N_\beta L_p + Y_v (N_p L_r - L_p N_r) + N_p L_\beta - \frac{g}{U_0}\left(L_\beta + \frac{I_{xz}}{I_{xx}} N_\beta\right) \tag{5-6}$$

$$E = \frac{g}{U_0} (L_\beta N_r - N_\beta L_r) \tag{5-7}$$

Equation 5-2 is referred to as the lateral-directional stability quartic. The roots in this case for any conventional vehicle are two real negatives or positives which are associated with two convergent or divergent motions and one pair of complex roots associated with an oscillatory motion.

The overall lateral directional motion of the vehicle is a superposition or a combination of all three modes.

The complex pair of roots correspond to the as named Dutch Roll oscillation, while the large real root corresponds to the Roll Subsidence mode and the small root to the Spiral mode.

For the numerical example given, we find:

$$A = 0.9975 \tag{5-8}$$

$$B = 1.867 \tag{5-9}$$

$$C = 3.6841 \tag{5-10}$$

$$D = 6.2637 \tag{5-11}$$

$$E = -0.0086 \tag{5-12}$$

and the characteristic polynomial is:

$$0.9975\, s^4 + 1.867\, s^3 + 3.6841\, s^2 + 6.2657\, s - 0.0085 = 0 \tag{5-13}$$

which has roots:

$$-0.0465 \pm j1.8784 \tag{5-14}$$

$$+0.014 \tag{5-15}$$

$$-1.7801 \tag{5-16}$$

The pair of roots, $-0.0465 \pm j1.8784$, correspond to the Dutch Roll mode, while the root $+0.0014$ corresponds to the Spiral mode and the root $-1.7801$ to the Roll Subsidence mode.

Further, it is more convenient to put the characteristic polynomial in the form:

$$(s^2 + 2\zeta_{DR}\,\omega n_{DR}\, s + \omega n^2_{DR})(s - r_s)(s - r_{RS}) \tag{5-17}$$

so that the natural frequency, the damping and the real roots are obtained by inspection while the damped frequency of the Dutch Roll can be determined by:

$$\omega d_{DR} = \omega n_{DR} (1 - \zeta^2_{DR})^{1/2} \tag{5-18}$$

The root plotting in the complex plane will appear as in Figure 5-1.

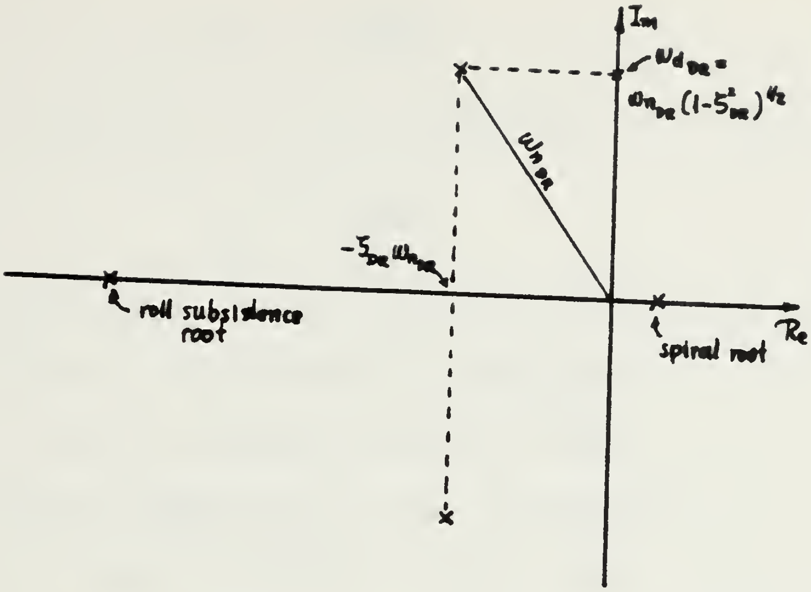

Im
$\omega_{d_{DR}} = \omega_{n_{DR}}(1-\zeta^2_{DR})^{1/2}$
$\omega_{n_{DR}}$
$-\zeta_{DR}\omega_{n_{DR}}$
roll subsidence root
spiral root
Re

Figure 5-1

Lateral-Directional Roots and Characteristic Quantities in the Complex Plane

The imaginary roots of the characteristic polynomial can be written:

$$-\zeta_{DR}\,\omega n_{DR} \pm j\,\omega_{d\,DR} \qquad (5\text{-}19)$$

or

$$-\zeta_{DR}\,\omega n_{DR} + j\,\omega n_{DR}\,(1-\zeta^2_{DR})^{1/2} \qquad (5\text{-}20)$$

Other quantities of importance in the lateral-directional dynamics are: the Dutch Roll period denoted by $T_{DR}$ given by:

$$T_{DR} = 2\pi/\omega_{d_{DR}} \qquad (5\text{-}21)$$

the half amplitude for the Dutch Roll given by:

$$t_{1/2_{DR}} = \frac{0.6931}{\zeta_{DR}\,\omega n_{DR}} \qquad (5\text{-}22)$$

the half amplitude for the Roll Subsidence mode (if stable)

$$t_{1/2_{RS}} = \frac{0.6931}{r_{RS}} \qquad (5\text{-}23)$$

and the half amplitude for the Spiral mode (if stable)

$$t_{1/2_{S}} = \frac{0.6931}{r_{S}} \qquad (5\text{-}24)$$

For our example, the characteristic polynomial of Equation 5-13 can be factored in the form:

$$(s^2 + 0.093\,s + 3.5305)(s - 0.0014)(s + 1.7801) \qquad (5\text{-}25)$$

From this by inspection we can determine:

$$\omega n_{DR} = (3.5305)^{1/2} = 1.879 \quad \text{rad/sec} \qquad (5\text{-}26)$$

$$\zeta_{DR} = 0.093/2 \cdot 1.879 = 0.0247 \qquad (5\text{-}27)$$

$$t_{1/2_{DR}} = 0.6931/0.0247 \cdot 1.879 = 14.9051 \text{ sec} \qquad (5\text{-}28)$$

$$t_{1/2_{RS}} = 0.6931/1.7801 = 0.3894 \text{ sec} \qquad (5\text{-}29)$$

Many of the basic ideas involving longitudinal stability, also apply to lateral and directional stability. Lateral and directional stability are interrelated due to cross coupling effects of roll and yaw motion, that give rise to the three modes of motion discussed so far.

In the Spiral mode, the airframe seems directionally nearly stable, with no sideslip but it is excited in a banked turn. The unstable mode is found usually in large finned airplanes with no dihedral, so that the side forces developed tend to turn the plane into the relative wind but since the outer wing travels faster, generates more lift and the plane will roll to still a higher bank angle. The flight of the

airplane is a heavy banked turn of ever-decreasing radius, i.e. a tightening spiral. The stable mode is found in airplanes with dihedral where the flight is a slight banked turn of gradually increasing radius. The spiral mode is illustrated in Figure 5-2.

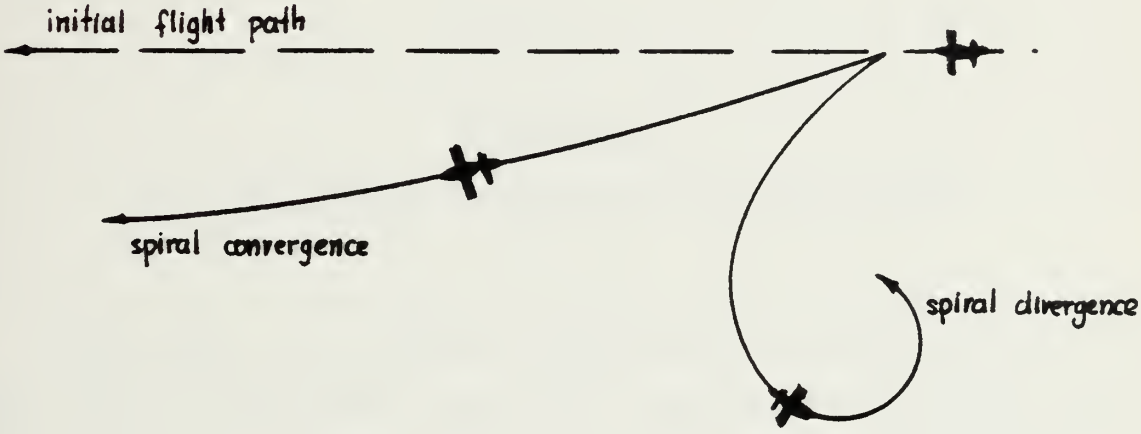


Figure 5-2

Spiral Stable and Unstable Modes

In the Roll Subsistence mode the airframe seems laterally nearly stable with no bank but it is excited in a sideslip turn. The unstable mode is present when side forces generate yawing moments that continue to increase the initial sideslip disturbance. This condition may continue until the airplane is broadside to the relative wind. Airplanes with successive weathercock stability, usually have this mode stable where an initial sideslip tends to die out. The Roll Subsistence mode is illustrated in Figure 5-3.

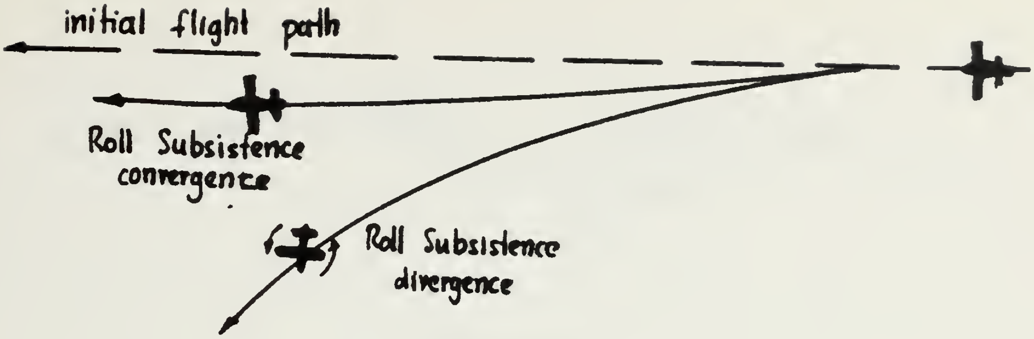


Figure 5-3

Roll Subsistence Stable and Unstable Modes

Dutch Roll is a motion which has the characteristics of the Spiral convergent and Roll Subsistence divergent mode. It is found in airplanes with strong lateral stability but weak directional. If a sideslip disturbance occurs and the plane is yawing in one direction, rolling moments are developed in a counter motion, i.e. yaw to the left is causing roll to the right and the plane wags its tail from side to side. Dutch Roll mode is illustrated in Figure 5-4.

The damped frequency of the Dutch Roll is indicated by the imaginary part of the complex root, while the real part indicates the damping. To reduce the effects of Dutch Roll, an increase in the directional and a decrease in the lateral stability is required.

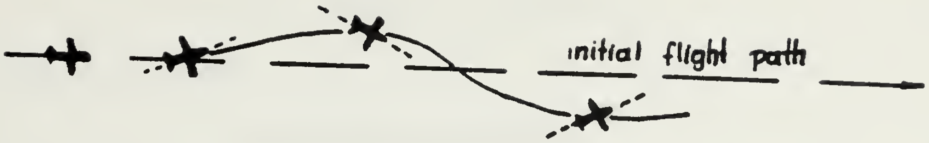


Figure 5-4

Dutch Roll Mode

## 5.3 DUTCH ROLL, SPIRAL, AND ROLL SUBSISTENCE MODE APPROXIMATIONS

By analogy with the longitudinal short period and phugoid approximations, we can reduce the degrees of freedom to obtain Dutch Roll Spiral and Roll Subsistence mode approximations.

Dutch Roll is a motion which is characterized by negligible variations of roll and sum of the rolling moments nearly zero. The Dutch Roll quadratic is obtained from the matrix Equation 5-1, by: (a) eliminating the $\phi$, and p equations, and (b) setting all rolling moment derivatives to zero obtaining thus, the determinant:

$$\begin{vmatrix} s - Y_v & 1 \\ -N_\beta & s - N_r \end{vmatrix} \qquad (5\text{-}30)$$

The characteristic Dutch Roll quadratic comes out to be:

$$s^2 - (Y_v + N_r)\, s + (Y_v N_r + N_\beta) = 0 \qquad (5\text{-}31)$$

For the example given, the characteristic Dutch Roll quadratic is

$$s^2 + 0.1786\,s + 3.5579 = 0 \tag{5-32}$$

which has the roots

$$-0.0893 \pm j1.8841 \tag{5-33}$$

These roots approximate the exact Dutch Roll roots.

From Equation 5-31 it is clear that the frequency of the Dutch Roll is heavily dependent on the weathercock derivative $N_\beta$ and the damping on the yaw damping derivative $N_r$ and the $Y_v$ derivative.

All the derivatives involved in the Dutch Roll mode are dependent mainly from the vertical tail, and secondarily, on the wings and fuselage. Thus, proper care has to be taken for the vertical tail as far as the Dutch Roll specifications. It is usually required that Dutch Roll oscillations should be damped to half amplitude within one cycle.

A first approximation to the Spiral and Roll Subsistence mode is obtained by expanding the determinant, Equation 5-30, by the roll damping term $(s-L_p)$ or $s(s-L_p)\phi$. The determinant takes the form of:

$$\begin{vmatrix} s-Y_v & 0 & 1 \\ 0 & s(s-L_p) & 0 \\ -N_\beta & 0 & s-N_r \end{vmatrix} \tag{5-34}$$

The characteristic equation comes out to be:

$$s(s-L_p)\left[s^2-(Y_v+N_r)s+(Y_vN_r+N_\beta)\right]=0 \qquad (5\text{-}35)$$

Here, the free $s$ corresponds to the Spiral mode, the $(s-L_p)$ corresponds to the Roll Subsistence mode, and the quadratic to the Dutch Roll mode.

By this approximation, for the given example, the Spiral root is zero, the Roll Subsistence root is -1.695 and the Dutch Roll mode as previously -0.0893 ± j1.8841.

A more precise approximation for the Spiral and Roll Subsistence modes, is obtained by expanding the determinant

$$\begin{vmatrix} 0 & -g/U_0 & 1 \\ -L_\beta & s(s-L_p) & -L_r \\ -N_\beta & s-N_p & s-N_r \end{vmatrix} \qquad (5\text{-}36)$$

This lead to

$$N_\beta s^2+\left(N_pL_\beta-N_\beta L_p-\frac{g}{U_0}L_\beta\right)s+\frac{g}{U_0}\left(L_\beta N_r-N_\beta L_r\right)=0 \qquad (5\text{-}37)$$

or

$$s^2+\left(\frac{N_pL_\beta}{N_\beta}-L_p-\frac{g}{U_0}\frac{L_\beta}{N_\beta}\right)s+\frac{g}{U_0}\left(\frac{L_\beta N_r}{N_\beta}-L_r\right)=0 \qquad (5\text{-}38)$$

For the given example one obtain the characteristic polynomial

$$s^2+1.7639\,s-0.0024=0 \qquad (5\text{-}39)$$

which has roots

$$0.0014,\ -1.7653 \qquad (5\text{-}40)$$

These roots approximate the exact Spiral and Roll Subsistence roots.

## 5.4 SOLUTION TO THE LATERAL-DIRECTIONAL EQUATIONS

Following the same approach as in the preceeding chapter, we first try to arrange the lateral-directional equations in the form

$$\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) \qquad (5\text{-}41)$$

For convenience, the untransformed equations are replicated below:

$$\dot{\beta} - (Y_v)\beta + r - (g/U_0)\phi = (Y_\delta^*)\delta \qquad (5\text{-}42)$$

$$-(L_\beta)\beta + \dot{p} - (L_p)p - (I_{xz}/I_{xx})\dot{r} - (L_r)r = (L_\delta)\delta \qquad (5\text{-}43)$$

$$-(N_\beta)\beta - (I_{xz}/I_{zz})\dot{p} - (N_p)p + \dot{r} - (N_r)r = (N_\delta)\delta \qquad (5\text{-}44)$$

$$\dot{\phi} = p \qquad (5\text{-}45)$$

$$\dot{\psi} = r \qquad (5\text{-}46)$$

Equation 5-42 is written

$$\dot{\beta} = (Y_v)\beta - r + (g/U_0)\phi + (Y_\zeta^*)\zeta \qquad (5\text{-}47)$$

Equation 5-43 is written

$$\dot{p} = (L_\beta)\beta + (L_p)p + (I_{xz}/I_{xx})\dot{r} + (L_r)r + (L_\xi)\xi + (L_\zeta)\zeta \qquad (5\text{-}48)$$

Equation 5-44 is written

$$\dot{r} = (N_\beta)\beta + (I_{xz}/I_{zz})\dot{p} + (N_p)p + (N_r)r + (N_\xi)\xi + (N_\zeta)\zeta \qquad (5\text{-}49)$$

In Equations 5-47, 5-48 and 5-49, the control inputs are expanded in terms of aileron ($\xi$) and rudder ($\zeta$) deflections, as indicated by Equation 3-143 neglecting the dotted derivatives $L_{\dot{\xi}}, L_{\dot{\zeta}}$.

Substituting Equation 5-49 into Equation 5-48 we obtain

$$\dot{p} = \frac{L_\beta + {}^{I_{xz}}/_{I_{xx}} N_\beta}{1 - I^2_{xz}/I_{xx}I_{zz}} \beta + \frac{L_p + {}^{I_{xz}}/_{I_{xx}} N_p}{1 - I^2_{xz}/I_{xx}I_{zz}} p + \frac{L_r + {}^{I_{xz}}/_{I_{xx}} N_r}{1 - I^2_{xz}/I_{xx}I_{zz}} r$$

$$+ \frac{L_\xi + {}^{I_{xz}}/_{I_{xx}} N_\xi}{1 - I^2_{xz}/I_{xx}I_{zz}} \xi + \frac{L_\zeta + {}^{I_{xz}}/_{I_{xx}} N_\zeta}{1 - I^2_{xz}/I_{xx}I_{zz}} \zeta \qquad (5\text{-}50)$$

Similarly substituting Equation 5-48 into Equation 5-49 we obtain

$$\dot{r} = \frac{N_\beta + {}^{I_{xz}}/_{I_{zz}} L_\beta}{1 - I^2_{xz}/I_{xx}I_{zz}} \beta + \frac{N_p + {}^{I_{xz}}/_{I_{zz}} L_p}{1 - I^2_{xz}/I_{xx}I_{zz}} p + \frac{N_r + {}^{I_{xz}}/_{I_{zz}} L_r}{1 - I^2_{xz}/I_{xx}I_{zz}} r$$

$$+ \frac{N_\xi + {}^{I_{xz}}/_{I_{zz}} L_\xi}{1 - I^2_{xz}/I_{xx}I_{zz}} \xi + \frac{N_\zeta + {}^{I_{xz}}/_{I_{zz}} L_\zeta}{1 - I^2_{xz}/I_{xx}I_{zz}} \zeta \qquad (5\text{-}51)$$

We now define for convenience

$$L^*_\beta = \frac{L_\beta + {}^{I_{xz}}/_{I_{xx}} N_\beta}{1 - I^2_{xz}/I_{xx}I_{zz}} \qquad (5\text{-}52)$$

$$L^*_p = \frac{L_p + {}^{I_{xz}}/_{I_{xx}} N_p}{1 - I^2_{xz}/I_{xx}I_{zz}} \qquad (5\text{-}53)$$

etc., so that Equation 5-50 and 5-51 can be written

$$\dot{p} = (L^*_\beta)\beta + (L^*_p)p + (L^*_r)r + (L^*_\xi)\xi + (L^*_\zeta)\zeta \qquad (5\text{-}54)$$

$$\dot{r} = (N^*_\beta)\beta + (N^*_p)p + (N^*_r)r + (N^*_\xi)\xi + (N^*_\zeta)\zeta \qquad (5\text{-}55)$$

Collecting Equations 5-47, 5-54, 5-55, 5-45 and 5-46 in matrix form we obtain the state variable representation of the lateral directional dynamics model, the state variables being $\beta$, p, r, $\phi$ and $\psi$.

$$\begin{vmatrix} \dot{\beta} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \\ \dot{\psi} \end{vmatrix} = \begin{vmatrix} Y_v & 0 & -1 & g/U_0 & 0 & 0 \\ L^*_\beta & L^*_p & L^*_r & 0 & 0 & 0 \\ N^*_\beta & N^*_p & N^*_r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} \beta \\ p \\ r \\ \phi \\ \psi \end{vmatrix} + \begin{vmatrix} 0 & Y^*_\zeta \\ L^*_\xi & L^*_\zeta \\ N^*_\xi & N^*_\zeta \\ 0 & 0 \\ 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} \xi \\ \zeta \end{vmatrix} \qquad (5-56)$$

The solution that follows corresponds to the given example with initial conditions and control inputs indicated in the summary of specifications tables.

For the Dutch Roll approximation the following model may be used

$$\begin{vmatrix} \dot{\beta} \\ \dot{r} \\ \dot{\psi} \end{vmatrix} = \begin{vmatrix} Y_v & -1 & 0 \\ N_\beta & N_r & 0 \\ 0 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} \beta \\ r \\ \psi \end{vmatrix} + \begin{vmatrix} 0 & Y^*_\zeta \\ N_\xi & N_\zeta \\ 0 & 0 \end{vmatrix} \begin{vmatrix} \xi \\ \zeta \end{vmatrix} \qquad (5-57)$$

which includes all the parameters of the determinant, Equation 5-30, while for the Roll Subsistence and Spiral approximation

$$\begin{vmatrix} \dot{\beta} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \\ \dot{\psi} \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 & g/U_0 & 0 \\ L_\beta & L_p & L_r & 0 & 0 \\ N_\beta & N_p & N_r & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} \beta \\ p \\ r \\ \phi \\ \psi \end{vmatrix} + \begin{vmatrix} 0 & Y^*_\zeta \\ L_\xi & L_\zeta \\ N_\xi & N_\zeta \\ 0 & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} \xi \\ \zeta \end{vmatrix} \qquad (5-58)$$

which includes all the parameters of the determinant, Equation 5-36.

## 5.5 LATERAL-DIRECTIONAL AERODYNAMIC TRANSFER FUNCTIONS

Following a same approach as in the preceeding chapter, we can develope lateral-directional aerodynamix transfer functions.

For the lateral-directional model, the input variables were the aileron control deflection ($\xi$) and the rudder control deflection ($\zeta$) while the output variables were $\beta$, p, r, $\phi$ and $\psi$.

Thus

$$\frac{\beta(s)}{\delta(s)} = \frac{\begin{vmatrix} Y_\delta^* & 0 & 1 & -g/U_0 & 0 \\ L_\delta & s-L_p & -s\,{}^{I_{xz}}/_{I_{xx}}-L_r & 0 & 0 \\ N_\delta & -s\,{}^{I_{xz}}/_{I_{zz}}-N_p & s-N_r & 0 & 0 \\ 0 & 1 & 0 & -s & 0 \\ 0 & 0 & 1 & 0 & -s \end{vmatrix}}{s\left[As^4+Bs^3+Cs^2+Ds+E\right]}$$

$$= \frac{A_\beta s^3+B_\beta s^2+C_\beta s+D_\beta}{As^4+Bs^3+Cs^2+Ds+E} \tag{5-59}$$

where

$$A_\beta = Y_\delta^*\left(1-I_{xz}^2/I_{xx}I_{zz}\right) \tag{5-60}$$

$$B_\beta = -Y_\delta^*\left[L_p+N_r+{}^{I_{xz}}/_{I_{xx}}N_p+{}^{I_{xz}}/_{I_{zz}}L_r\right]-{}^{I_{xz}}/_{I_{zz}}L_\delta-N_\delta \tag{5-61}$$

$$C_\beta = Y_\delta^*\left(L_pN_r-N_pL_r\right)+N_\delta L_p-L_\delta N_p+{}^{g}/_{U_0}\left(L_\delta+{}^{I_{xz}}/_{I_{xx}}N_\beta\right) \tag{5-62}$$

$$D_\beta = {}^{g}/_{U_0}\left(N_\delta L_r-L_\delta N_r\right) \tag{5-63}$$

This transfer function usually takes the form

$$\frac{\beta(s)}{\delta(s)} = \frac{A_\beta (s+z_{\beta_1})(s+z_{\beta_2})(s+z_{\beta_3})}{As^4+Bs^3+Cs^2+Ds+E} \tag{5-64}$$

Other lateral-directional transfer functions of interest are

$$\frac{\phi(s)}{\delta(s)} = \frac{A_\phi s^2 + B_\phi s + C_\phi}{As^4+Bs^3+Cs^2+Ds+E} \tag{5-65}$$

where

$$A_\phi = L_\delta + {}^{I_{xz}}/_{I_{xx}} N_\delta \tag{5-66}$$

$$B_\phi = Y_\delta^* \left(L_\beta + {}^{I_{xz}}/_{I_{xx}} N_\beta\right) - L_\delta (N_r+Y_v) + N_\delta \left(L_r - {}^{I_{xz}}/_{I_{xx}} Y_v\right) \tag{5-67}$$

$$C_\phi = Y_\delta^* (L_r N_\beta - L_\beta N_r) + L_\delta (Y_v N_r + N_\beta) - N_\delta (L_\beta + Y_v L_r) \tag{5-68}$$

and takes the form

$$\frac{\phi(s)}{\delta(s)} = \frac{A_\phi (s^2 + 2\zeta_\phi \omega_\phi s + \omega_\phi^2)}{As^4+Bs^3+Cs^2+Ds+E} \tag{5-69}$$

Also

$$\frac{r(s)}{\delta(s)} = \frac{A_r s^3 + B_r s^2 + C_r s + D_r}{As^4+Bs^3+Cs^2+Ds+E} \tag{5-70}$$

where

$$A_r = N_\delta + {}^{I_{xz}}/_{I_{zz}} L_\delta \tag{5-71}$$

$$B_r = Y_\delta^* \left(N_\beta + {}^{I_{xz}}/_{I_{zz}} L_\beta\right) + L_\delta \left(N_p - {}^{I_{xz}}/_{I_{zz}} Y_v\right) - N_\delta (Y_v + L_p) \tag{5-72}$$

$$C_r = Y_\delta^* (L_\beta N_p - N_\beta L_p) - L_\delta Y_v N_p + N_\delta Y_v L_p \tag{5-73}$$

$$D_r = {}^{g}/_{U_0} (L_\delta N_\beta - N_\delta L_\beta) \tag{5-74}$$

and takes the following forms

$$\frac{r(s)}{\delta(s)} = \frac{A_r (s+z_{r_1})(s^2 + 2\zeta_r \omega_r s + \omega_r^2)}{As^4+Bs^3+Cs^2+Ds+E} \tag{5-75}$$

or

$$\frac{r(s)}{\delta(s)} = \frac{A_r(s+z_{r_1})(s+z_{r_2})(s+z_{r_3})}{As^4+Bs^3+Cs^2+Ds+E} \qquad (5\text{-}76)$$

It is further desirable to include the transfer function relating the acceleration on the y-axis to the control input. The acceleration on the y-axis is given from the original V-equation, Equation 2-135, which for =0 becomes

$$a_y = \dot{v} + U_0 r - g\phi \qquad (5\text{-}77)$$

and the corresponding transfer function is:

$$\frac{a_y(s)}{\delta(s)} = \frac{A_a s^4 + B_a s^3 + C_a s^2 + D_a s + E_a}{As^4+Bs^3+Cs^2+Ds+E} \qquad (5\text{-}78)$$

where

$$A_a = Y_\delta\left(1 - I_{xz}^2/I_{xx}I_{zz}\right) \qquad (5\text{-}79)$$

$$B_a = -Y_\delta\left(L_p + N_r + I_{xz}/I_{xx}\, N_p + I_{xz}/I_{zz}\, L_p\right) \qquad (5\text{-}80)$$

$$C_a = Y_\delta\left(N_\beta + L_p N_r - N_p L_r + I_{xz}/I_{zz}\, L_\beta\right) - U_0 Y_v\left(I_{xz}/I_{zz}\, L_\delta + N_\delta\right) \qquad (5\text{-}81)$$

$$D_a = Y_\delta\left[N_p L_\beta - \frac{g}{U_0} L_\beta - N_\beta\left(L_p + g/U_0\, I_{xz}/I_{xx}\right)\right] + U_0 Y_v\left[L_\delta\left(\frac{g}{U_0} - N_p\right) + N_\delta\left(\frac{g}{U_0}\frac{I_{xz}}{I_{xx}} + L_p\right)\right] \qquad (5\text{-}82)$$

$$E_a = g\left[Y_\beta^*\left(L_\beta N_r - N_\beta L_r\right) + Y_v\left(N_\delta L_r - L_\delta N_r\right)\right] \qquad (5\text{-}83)$$

and takes the form

$$\frac{a_y(s)}{\delta(s)} = \frac{A_a(s+z_{a_1})(s+z_{a_2})(s+z_{a_3})(s+z_{a_4})}{As^4+Bs^3+Cs^2+Ds+E} \qquad (5\text{-}84)$$

## 5.6 LATERAL-DIRECTIONAL EQUATIONS IN NON-DIMENSIONAL SYSTEMS

In the non-dimensional body axes system the lateral-directional equations of motion are [Ref. 2]:

$$\begin{vmatrix} 2\mu s - C_{y_\beta} & -sC_{y_p} - C_{L_0} & 2\mu - C_{y_r} \\ -C_{\ell_\beta} & s^2 \iota_A - sC_{\ell_p} & -s\iota_E - C_{\ell_r} \\ -C_{n_\beta} & -s^2 \iota_E - sC_{n_p} & s\iota_C - C_{n_r} \end{vmatrix} \cdot \begin{vmatrix} \beta \\ \phi \\ r \end{vmatrix} = \begin{vmatrix} 0 & C_{y_\zeta} \\ C_{\ell_\xi} & C_{\ell_\zeta} \\ C_{n_\xi} & C_{n_\zeta} \end{vmatrix} \begin{vmatrix} \xi \\ \zeta \end{vmatrix} \qquad (5\text{-}85)$$

where the non-dimensional mass, inertia terms and time are expressed as

$$\mu = m/\rho S \ell \qquad (5\text{-}86)$$

$\ell$ being now the half of the wing span ($\ell = b/2$)

$$\iota_A = I_{xx}/\rho S \ell^3 \qquad (5\text{-}87)$$

$$\iota_C = I_{zz}/\rho S \ell^3 \qquad (5\text{-}88)$$

$$\iota_E = I_{xz}/\rho S \ell^3 \qquad (5\text{-}89)$$

$$t^* = \ell/U_0 \qquad (5\text{-}90)$$

In the non-dimensional wind axes system the lateral directional equations are [Ref. 3]:

$$\begin{vmatrix} -2s + C_{y_\beta} & 2s & C_L \\ \mu C_{\ell_\beta} & \frac{1}{2} s C_{\ell_r} & \frac{1}{2} s C_{\ell_p} - s^2 I_x \\ \mu C_{n_\beta} & \frac{1}{2} C_{n_r} - s I_z & \frac{1}{2} s C_{n_p} \end{vmatrix} \cdot \begin{vmatrix} \beta \\ \psi \\ \phi \end{vmatrix} = \begin{vmatrix} 0 & C_{y_\xi} \\ C_{\ell_\xi} & C_{\ell_\zeta} \\ C_{n_\xi} & C_{n_\zeta} \end{vmatrix} \cdot \begin{vmatrix} \xi \\ \zeta \end{vmatrix} \qquad (5\text{-}91)$$

where the non-dimensional mass, inertia terms and time are expressed as

$$\mu = m/\rho S b \qquad (5\text{-}92)$$

$$I_x = 2\,\frac{I_{xx}/m}{\mu b^2} \qquad (5\text{-}93)$$

$$I_z = 2\,\frac{I_{zz}/m}{\mu b^2} \qquad (5\text{-}94)$$

$$\tau = m/\rho S U_0 \qquad (5\text{-}95)$$

$$t^* = t/\tau \qquad (5\text{-}96)$$

## TABLE 5-1

## SPECIFICATIONS FOR COMPUTER SOLUTION OF THE LATERAL-DIRECTIONAL EQUATIONS DUE TO INITIAL CONDITIONS

A. Variables and Initial Conditions

$\beta = 0.05$
$P = 0.05$
$r = 0.075$
$\phi = 0.1$
$\psi = 0.2$
$t = 0.0$

B. Stability Derviatives and Constants

$Y_v = -0.0829$
$Y_\zeta = 7.656$

$L_\beta = -4.77$
$L_p = -1.695$
$L_r = 0.1776$
$L_\xi = 27.25$
$L_\zeta = 0.666$

$N_\beta = 3.55$
$N_p = -0.0025$
$N_r = -0.0957$
$N_\xi = -0.615$
$N_\zeta = -1.383$

$I_{xz}/I_{xx} = 0.0663$
$I_{xz}/I_{zz} = 0.0370$
$g = 32.714$
$\upsilon_o = 660.0$

C. Special Functions

$$Y_\zeta^* = Y_\zeta/U_o$$

$$D = 1 - (I_{xz}^2/I_{xx}I_{zz})$$

$$L_\beta^* = [L_\beta + (I_{xz}/I_{xx})N_\beta]/D \qquad N_\beta^* = [N_\beta + (I_{xz}/I_{zz})L_\beta]/D$$

$$L_p^* = [L_p + (I_{xz}/I_{xx})N_p]/D \qquad N_p^* = [N_p + (I_{xz}/I_{zz})L_p]/D$$

$$L_r^* = [L_r + (I_{xz}/I_{xx})N_r]/D \qquad N_r^* = [N_r + (I_{xz}/I_{zz})L_r]/D$$

$$L_\xi^* = [L_\xi + (I_{xz}/I_{xx})N_\xi]/D \qquad N_\xi^* = [N_\xi + (I_{xz}/I_{zz})L_\xi]/D$$

$$L_\zeta^* = [L_\zeta + (I_{xz}/I_{xx})N_\zeta]/D \qquad N_\zeta^* = [N_\zeta + (I_{xz}/I_{zz})L_\zeta]/D$$

D. Control Inputs

$\xi = 0.0$
$\zeta = 0.0$

E. Derivatives

$$\dot{\beta} = Y_v \beta - r + (g/U_0)\phi + Y^*_\zeta \zeta$$

$$\dot{p} = L^*_\beta \beta + L^*_p p + L^*_r r + L^*_\xi \xi + L^*_\zeta \zeta$$

$$\dot{r} = N^*_\beta \beta + N^*_p p + N^*_r r + N^*_\xi \xi + N^*_\zeta \zeta$$

$$\dot{\phi} = p$$

$$\dot{\psi} = r$$

F. Outputs
$\beta$, p, r, $\phi$, $\psi$ vs time at interval 0.0125
end calculation when time > 50.0

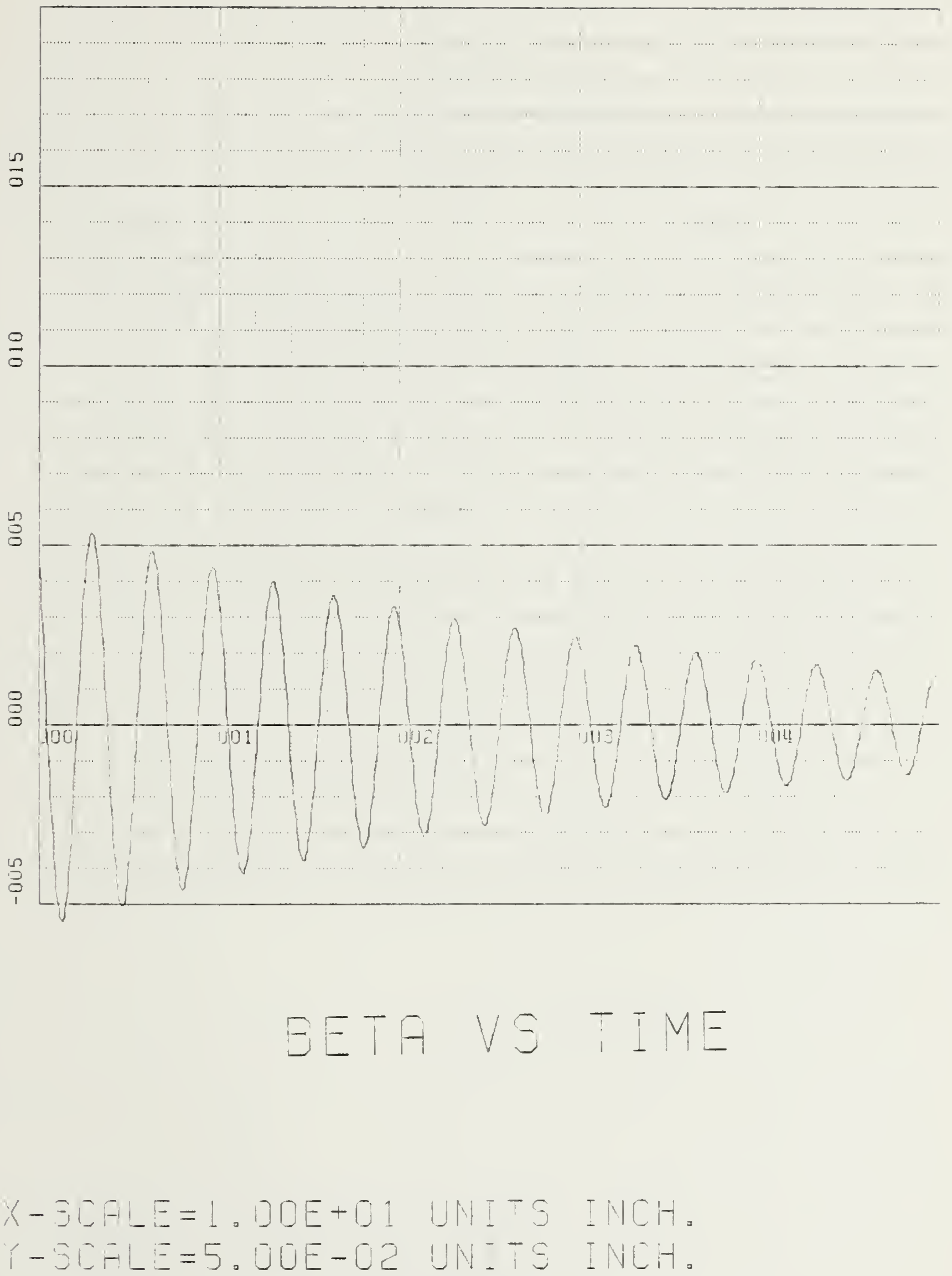


Figure 5-5

$\beta$-Sideslip Angle vs Time for Specifications of Table 5-1

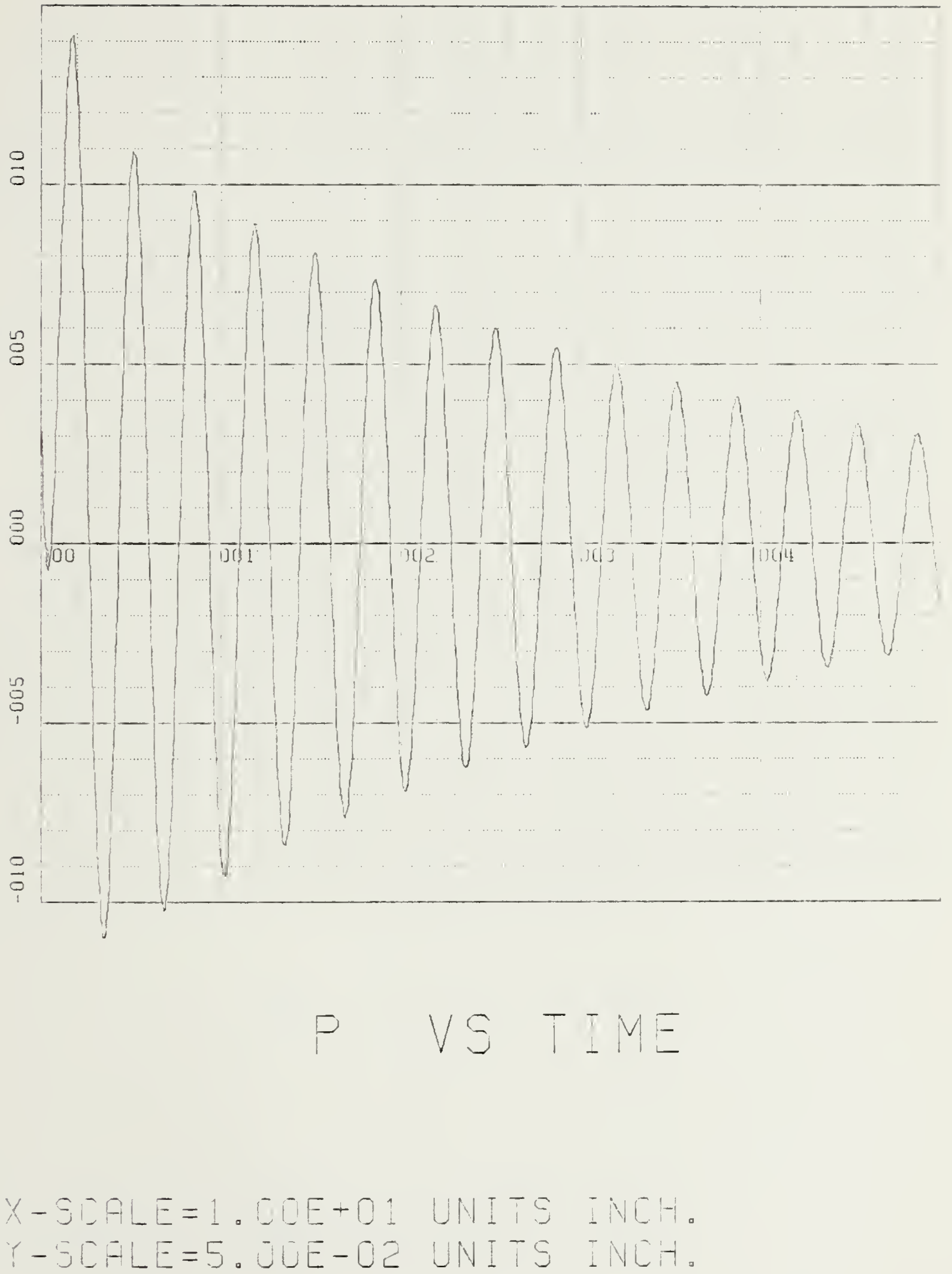


Figure 5-6

p-Roll Rate vs Time for Specifications of Table 5-1

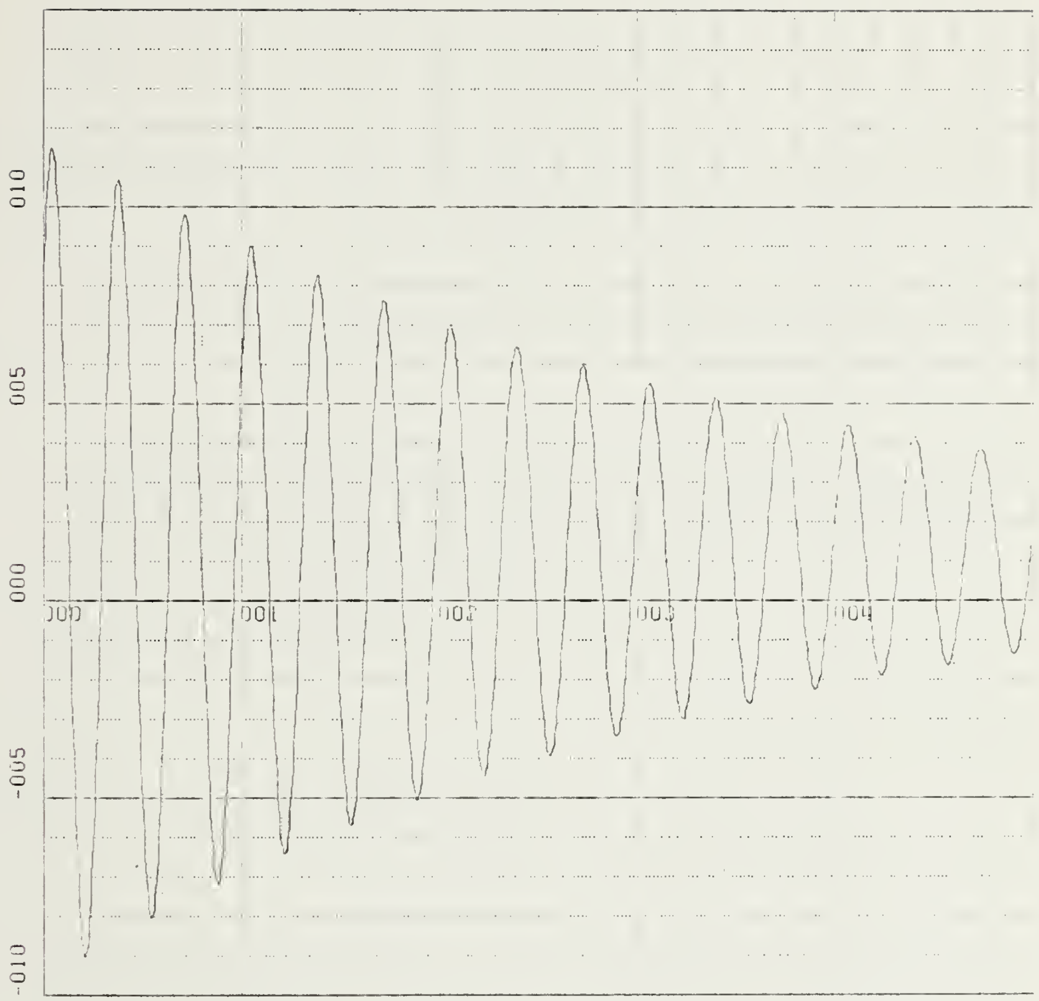


Figure 5-7

r-Yaw-Rate vs Time for Specifications of Table 5-1

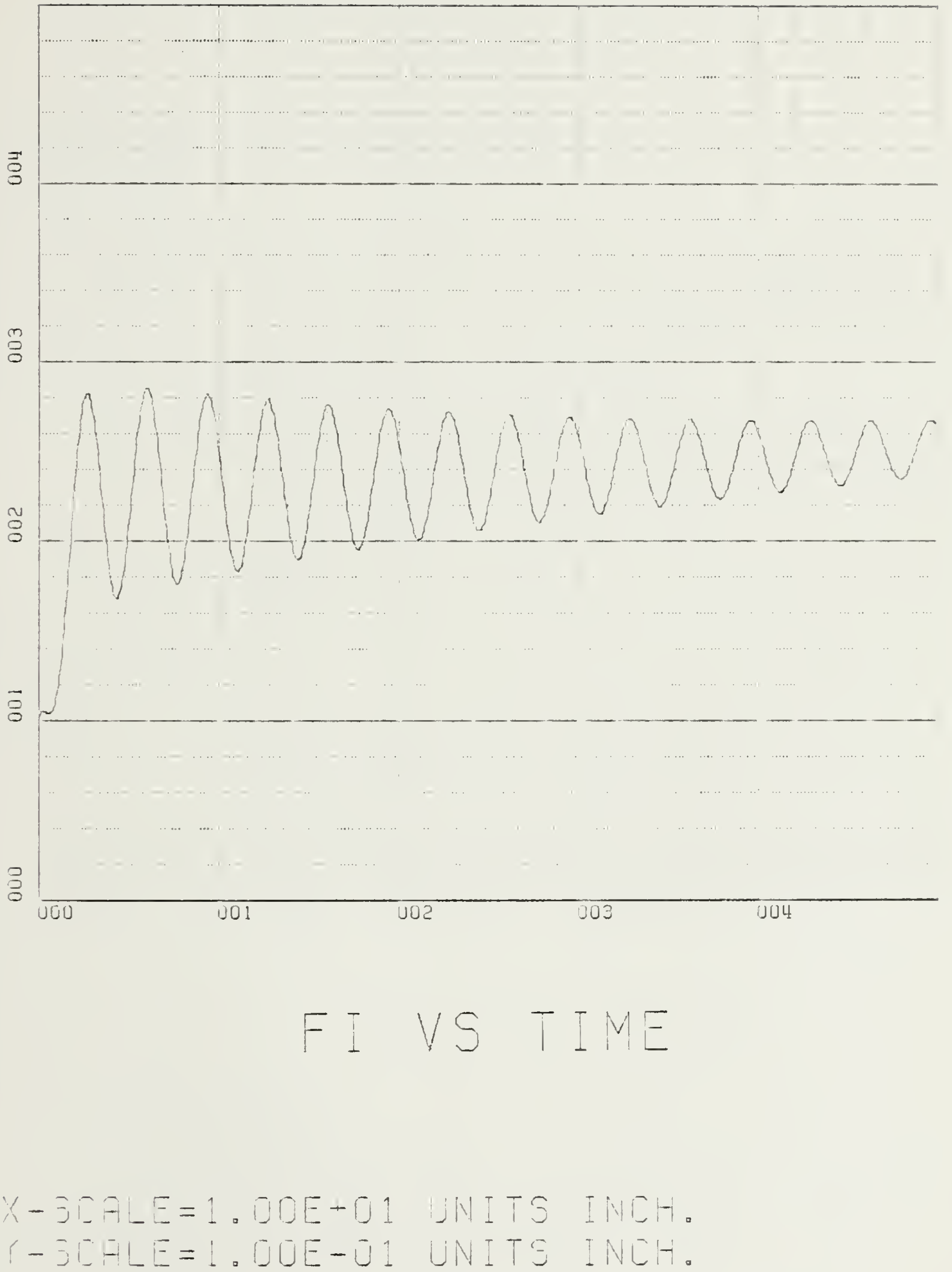


Figure 5-8

Φ-Bank Angle vs Time for Specifications of Table 5-1

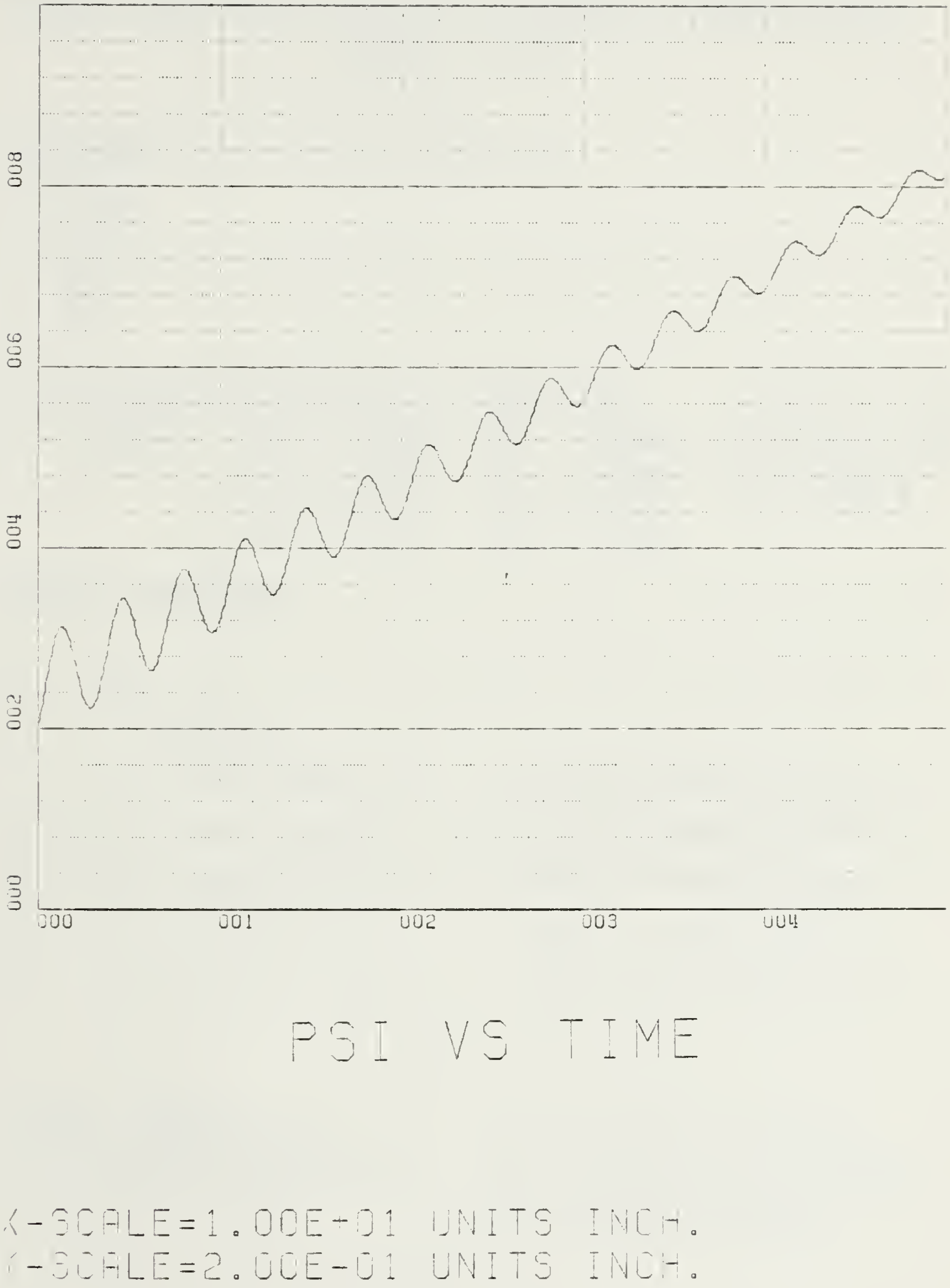


Figure 5-9

ψ-Yaw Angle vs Time for Specifications of Table 5-1

## TABLE 5-2

## SPECIFICATIONS FOR COMPUTER SOLUTION OF THE LATERAL-DIRECTIONAL EQUATIONS DUE TO INITIAL CONDITIONS AND CONTROL INPUTS

A. Variables and Initial Conditions

$\beta = 0.05$

$p = 0.05$

$r = 0.075$

$\Phi = 0.1$

$\psi = 0.2$

$t = 0.0$

B. Stability Derivatives and Constants

$Y_v = -0.0829$

$Y_\zeta = 7.656$

$L_\beta = -4.77$

$L_p = -1.695$

$L_r = 0.1776$

$L_\xi = 27.25$

$L_\zeta = 0.666$

$N_\beta = 3.55$

$N_p = -0.0025$

$N_r = -0.0957$

$N_\xi = -0.615$

$N_\zeta = -1.383$

$I_{xz}/I_{xx} = 0.0663$

$I_{xz}/I_{zz} = 0.0370$

$g = 32.174$

$U_o = 660.0$

C. Special Functions

$$Y^*_\zeta = Y_\zeta / U_o$$

$$D = 1 - (I_{xz}^2 / I_{xx} I_{zz})$$

$$L^*_\beta = [L_\beta + (I_{xz}/I_{xx}) N_\beta] / D \qquad N^*_\beta = [N_\beta + (I_{xz}/I_{zz}) L_\beta] / D$$

$$L^*_p = [L_p + (I_{xz}/I_{xx}) N_p] / D \qquad N^*_p = [N_p + (I_{xz}/I_{zz}) L_p] / D$$

$$L^*_r = [L_r + (I_{xz}/I_{xx}) N_r] / D \qquad N^*_r = [N_r + (I_{xz}/I_{zz}) L_r] / D$$

$$L^*_\xi = [L_\xi + (I_{xz}/I_{xx}) N_\xi] / D \qquad N^*_\xi = [N_\xi + (I_{xz}/I_{zz}) L_\xi] / D$$

$$L^*_\zeta = [L_\zeta + (I_{xz}/I_{xx}) N_\zeta] / D \qquad N^*_\zeta = [N_\zeta + (I_{xz}/I_{zz}) L_\zeta] / D$$

D. Control Inputs

$\xi = 0.01$ for $0<t<2.5$

$\xi = -0.01$ for $2.5<t<5.0$

$\xi = 0.0$ for $t>5.0$

$\zeta = 0.0$ for $0<t<15.0$

$\zeta = 0.1$ for $15.0<t<17.5$

$\zeta = -0.1$ for $17.5<t<20.0$

$\zeta = 0.0$ for $t>20.0$

TABLE 5-2 (Continued)

E. Derivatives

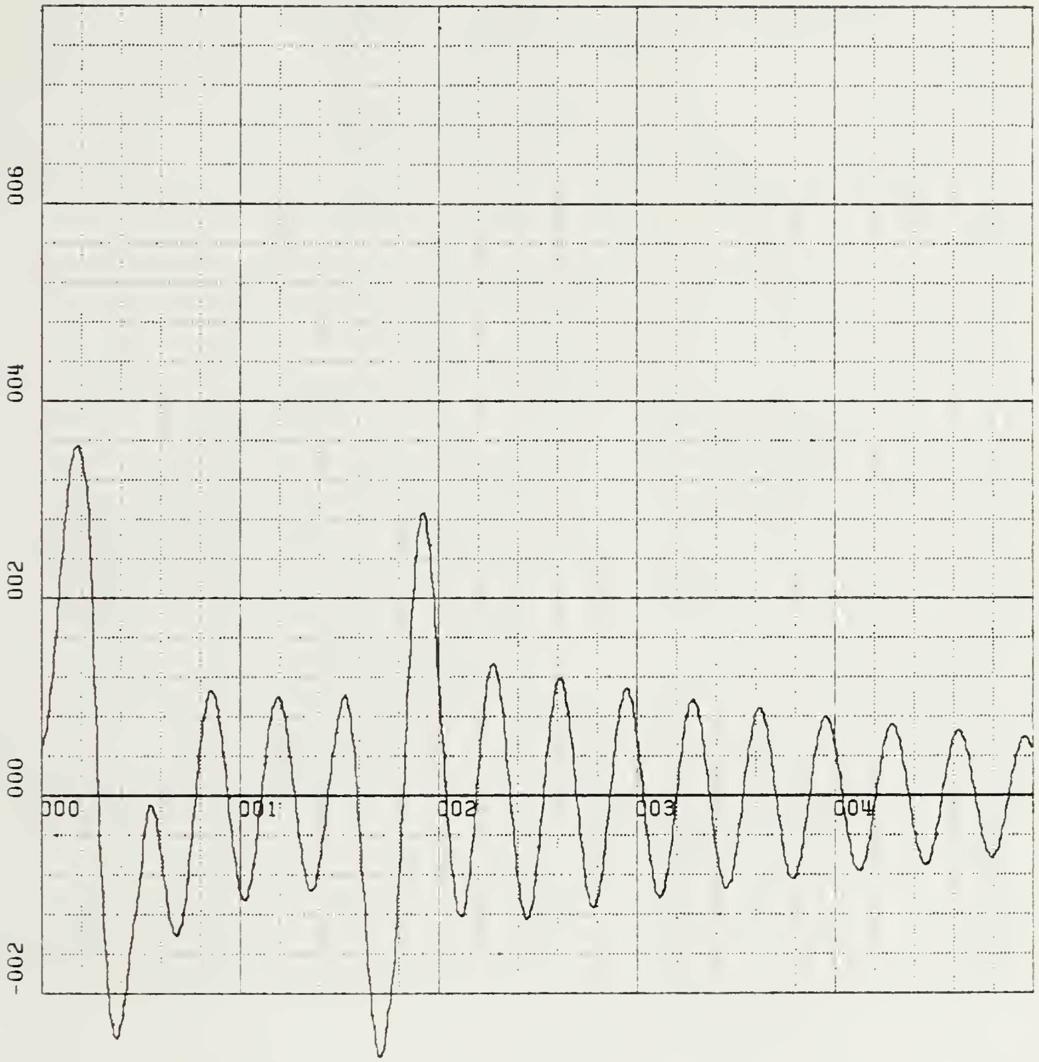
$$\dot{\beta} = Y_v \beta - r + (g/U_0)\phi + Y^*_\zeta \zeta \qquad \dot{\phi} = p$$

$$\dot{p} = L^*_\beta \beta + L^*_p p + L^*_r r + L^*_\xi \xi + L^*_\zeta \zeta \qquad \dot{\psi} = r$$

$$\dot{r} = N^*_\beta \beta + N^*_p p + N^*_r r + N^*_\xi \xi + N^*_\zeta \zeta$$

F. Outputs

$\beta$, p, r, $\phi$, $\psi$ vs time at interval 0.0125
end calculation when time > 50.0

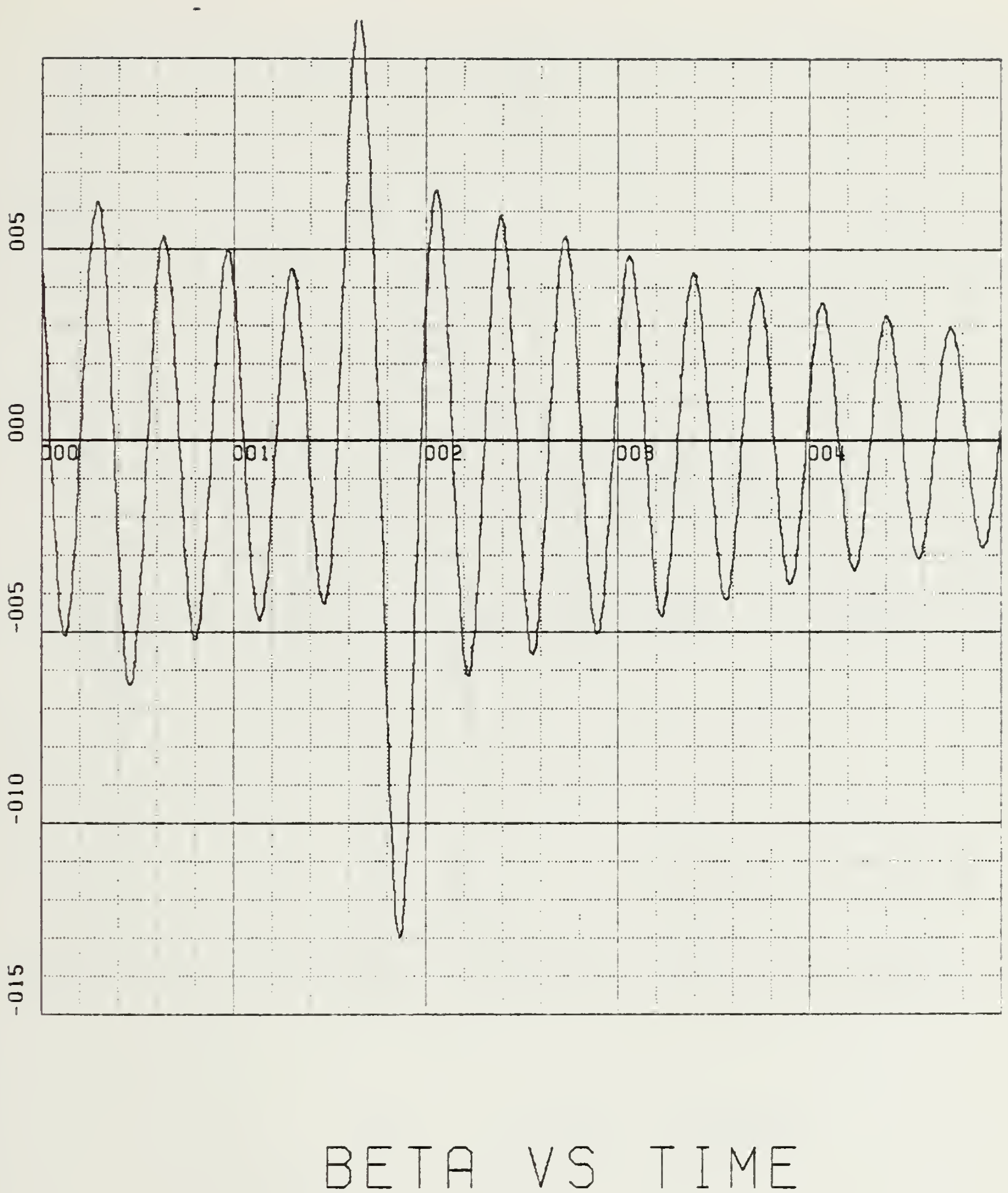


P VS TIME

X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=2.00E-01 UNITS INCH.

Figure 5-11

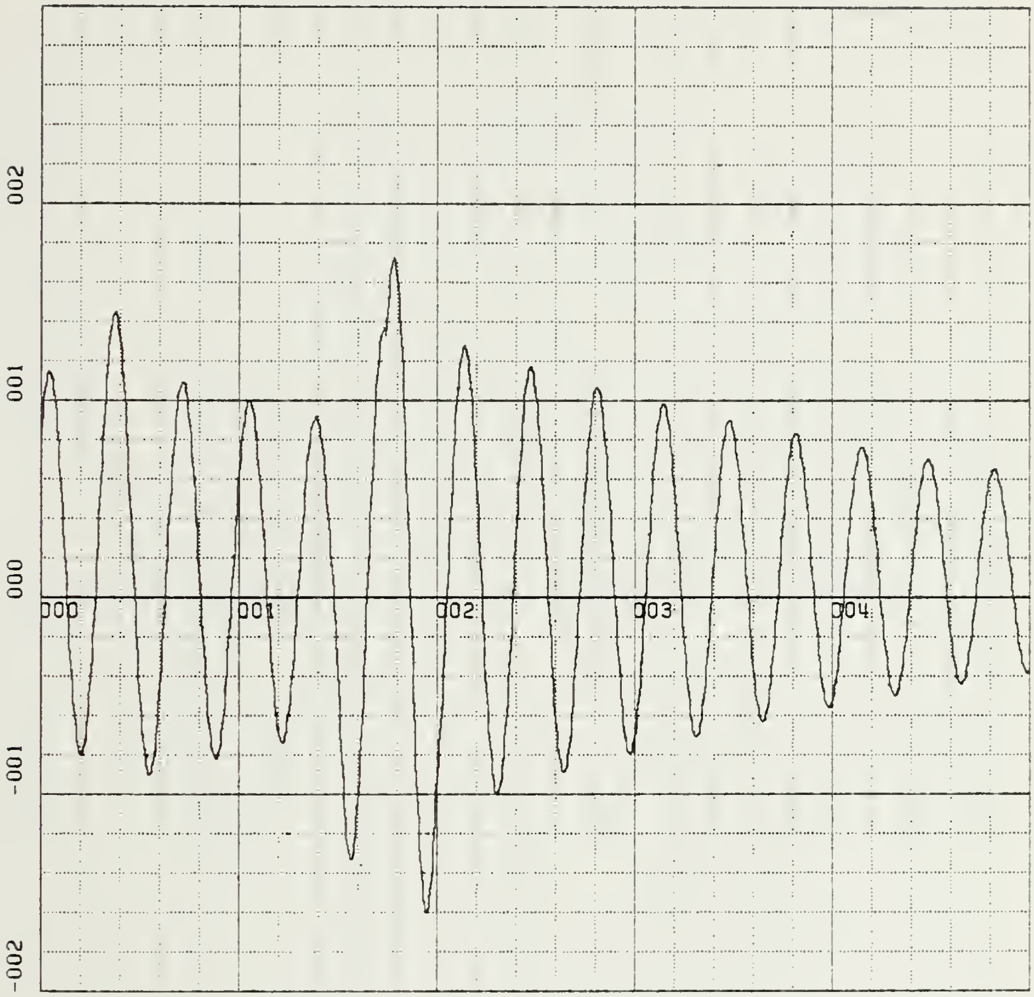
p-Roll Rate vs Time for Specifications of Table 5-2



X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=5.00E-02 UNITS INCH.

Figure 5-10

β-Sideslip Angle vs Time for Specifications of Table 5-2

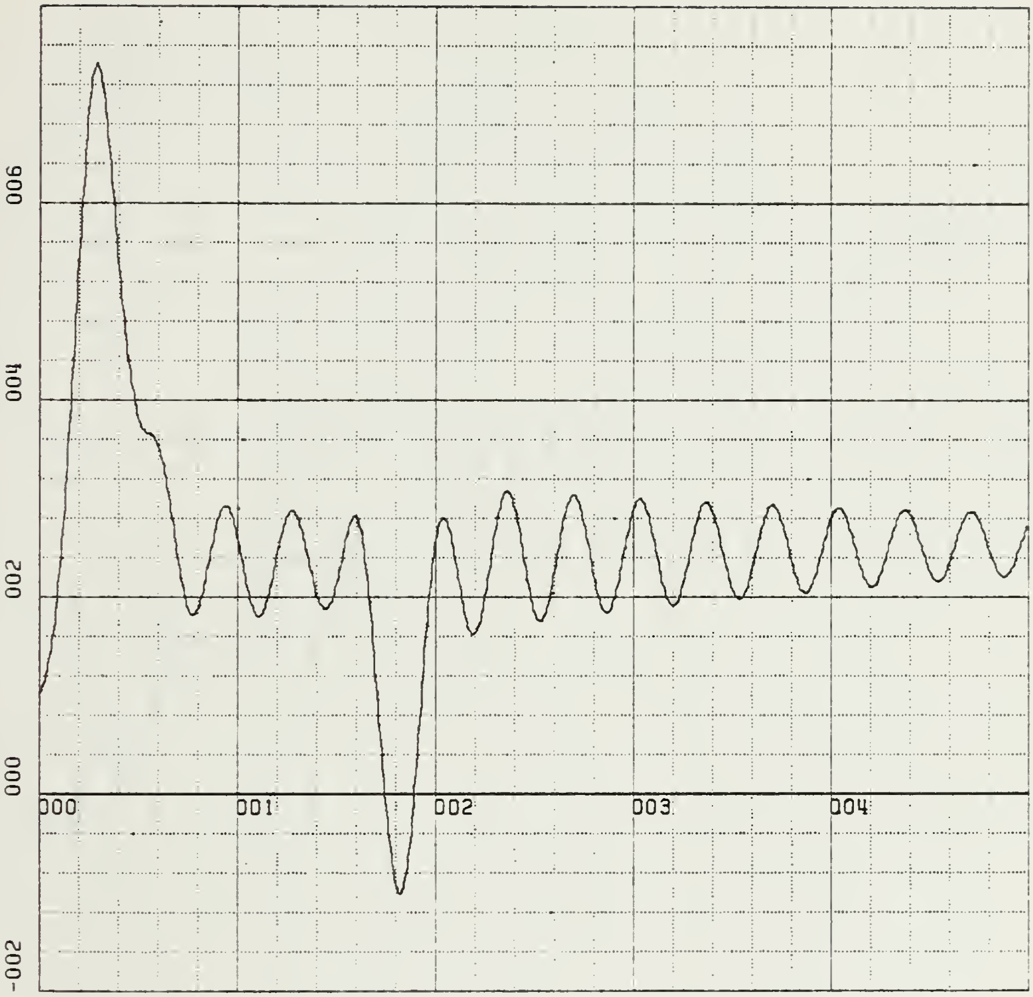


R VS TIME

X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=1.00E-01 UNITS INCH.

Figure 5-12

r-Yaw Rate vs Time for Specifications of Table 5-2



FI VS TIME

X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=2.00E-01 UNITS INCH.

Figure 5-13

$\phi$-Bank Angle vs Time for Specifications of Table 5-2

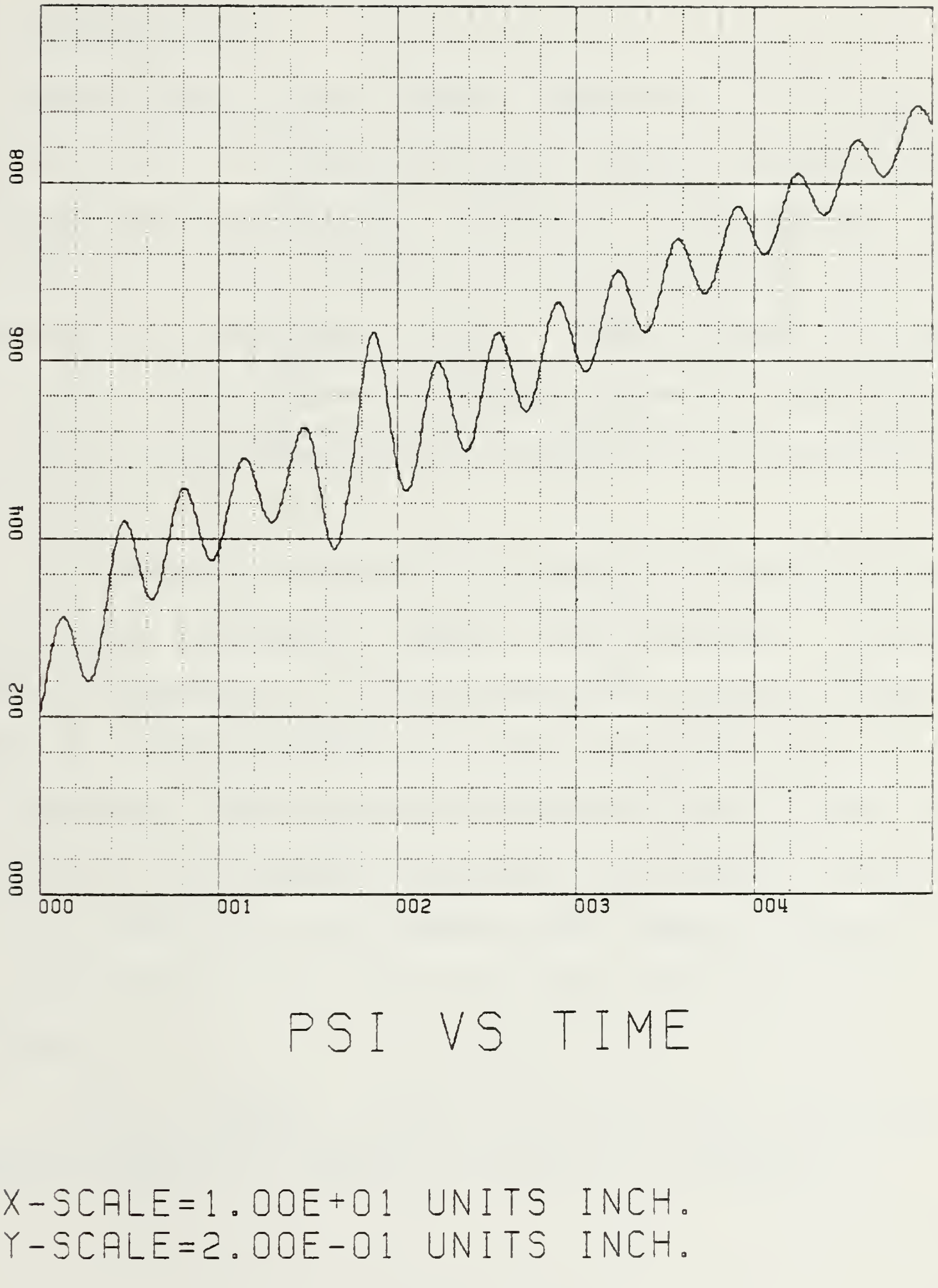


Figure 5-14

ψ -Yaw Angle vs Time for Specifications of Table 5-2

# CHAPTER 6

# STATE VARIABLE FEEDBACK LAW

## 6.1 INTRODUCTION

The automatic-flight control functions of an aircraft are provided by an automatic flight control system which is referred to as autopilot. Autopilot statilizes the dynamic response of the aircraft providing the best response to disturbances and control inputs. Today, almost all types of aircrafts, helicopters and missiles are equipped with autopilots with complication depending on the particular application.

The equipment comprising the autopilot includes sensors, controllers, estimators, computer and actuators. To damp the vehicle responses to disturbances and control inputs, the rates of the angular motion in roll, pitch and yaw are sensed by gyroscopic instruments and fed back to the flight control computer. The computer combines the feedback information with the desired pilot response and computes commands which are applied to the actuators that deflect the control surfaces.

In this chapter we will discuss the state variable feedback method as a means to control the vehicle.

## 6.2 THE FEEDBACK CONCEPT

The negative feedback attempts to modify the characteristic equation of the system so that one can get the specified by the requirements response.

Consider as an example that the roll motion of an aircraft is described by the equation

$$\dot{p} = L_p p + L_\delta \delta \tag{6-1}$$

The open loop transfer function is

$$\frac{p(s)}{\delta(s)} = \frac{L_\delta}{s - L_p} \tag{6-2}$$

Suppose that the p-variable is continuously measured and fed back as shown in Figure 6-1.

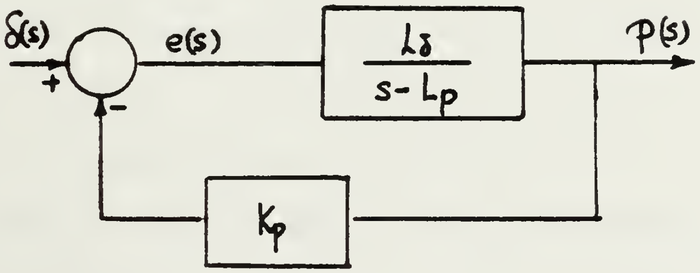


Figure 6-1

Simple Feedback System for Roll Stabilization

The closed loop transfer function is

$$\frac{p(s)}{\delta(s)} = \frac{L_\delta/(s-L_p)}{1+K_p L_\delta/(s-L_p)} = \frac{L_\delta}{s+(K_p L_\delta - L_p)} \tag{6-3}$$

It is clear that the single characteristic root changed from $L_p$ to $K_p \cdot L_\delta - L_p$. Usually $L_p$ is negative and $L_\delta$ positive. So the system will have a faster response indicated by the greater negative root of the characteristic polynomial.

In automatic control systems all the variables of the system are fed back with corresponding gains K's chosen to give a particular desired response.

In general, the system is described in matrix form as

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \tag{6-4}$$

with initial condition vector

$$\underline{x}(0) = \underline{x}_0 \tag{6-5}$$

Applying the feedback law

$$\underline{u} = -\underline{K}\,\underline{x} + \underline{u}_d \tag{6-6}$$

where

$$K = [K_1 \;\; K_2 \;\; K_3 \;\; \ldots \;\; K_n] \tag{6-7}$$

is the feedback gain matrix and $\underline{u}_d$ the desired input, one obtains the system

$$\dot{\underline{x}} = (\underline{A} - \underline{B}\,\underline{K})\,\underline{x} + \underline{B}\,\underline{u}_d \tag{6-8}$$

The matrix A-BK now determines the new characteristic roots of the system. Note that the characteristic roots of the system, Equation 6-4, are obtained by the determinant

$$|s\underline{I} - \underline{A}| = 0 \tag{6-9}$$

while for the system, Equation 6-8, by the determinant

$$|s\underline{I} - (\underline{A} - \underline{B}\underline{K})| = 0 \qquad (6\text{-}10)$$

Since the response of the system is determined by the location of the characteristic roots, any desired response will specify the characteristic polynomial of the system which if equated with Equation 6-10 the feedback gains are obtained.

Feeding back all the variables means that somehow all of them must be measured or estimated. Usually not all of the variables are measured but the ones that are not measured are estimated. For example, the pitch angle can be obtained by integration of the q-pitch rate.

$$\theta = \frac{1}{s} q \qquad (6\text{-}11)$$

Even unstable systems can be stabilized with state variable feedback provided that the system is controlable.

Most times the feedback gains are selected such that for the system to meet a performance criterion. In this case optical control methods are used to select the gains.

In the next pages a computer solution is shown of the given example for the longitudinal and lateral directional case using the state variable feedback concept. Specifications for computer solution are shown in Tables 6-1 and 6-2.

## TABLE 6-1

## SPECIFICATIONS FOR COMPUTER SOLUTION OF THE LONGITUDINAL EQUATIONS DUE TO INITIAL CONDITIONS, CONTROL INPUTS AND STATE VARIABLE FEEDBACK STABILIZATION

A. Variables and Initial Conditions

$\upsilon$ = 5.0 ft/sec
w = 2.5 ft/sec
q = 0.05 rad/sec
$\theta$ = 0.075 rad
h = 10.0 ft
t = 0.0

B. Stability Deriviatives and Constants

$X_\upsilon = -0.0097$
$X_w = 0.0016$
$X_\delta = 0.0$
$Z_\upsilon = -0.0955$
$Z_w = -1.43$
$Z_\delta = -69.8$
$M_\upsilon = 0.0$
$M_w = -0.0235$
$M_{\dot{w}} = -0.0013$
$M_q = -1.92$
$M_\delta = -26.10$
$g = 32.174$
$\upsilon_o = 660.0$

C. Special Functions

$M_\upsilon^* = M_\upsilon + M_{\dot{w}} Z_\upsilon$

$M_w^* = M_w + M_{\dot{w}} Z_w$

$M_q^* = M_q + M_{\dot{w}} U_0$

$M_\delta^* - M_\delta + M_{\dot{w}} Z_\delta$

D. Control Inputs

= 0.01+ f for 0<t<10.0
= -0.01+ f for 10.0<t<20.0
= 0.0 + f for t>20.0

$f = -0.005\,\upsilon - 0.0005\,w + 0.022\,q + 0.003\,\theta + 0.0000001\,h$

E. Derivatives

$\dot{\upsilon} = X_\upsilon \upsilon + X_w w - g\theta + X_\delta \delta$

$\dot{w} = Z_\upsilon \upsilon + Z_w w + U_0 q + Z_\delta \delta$

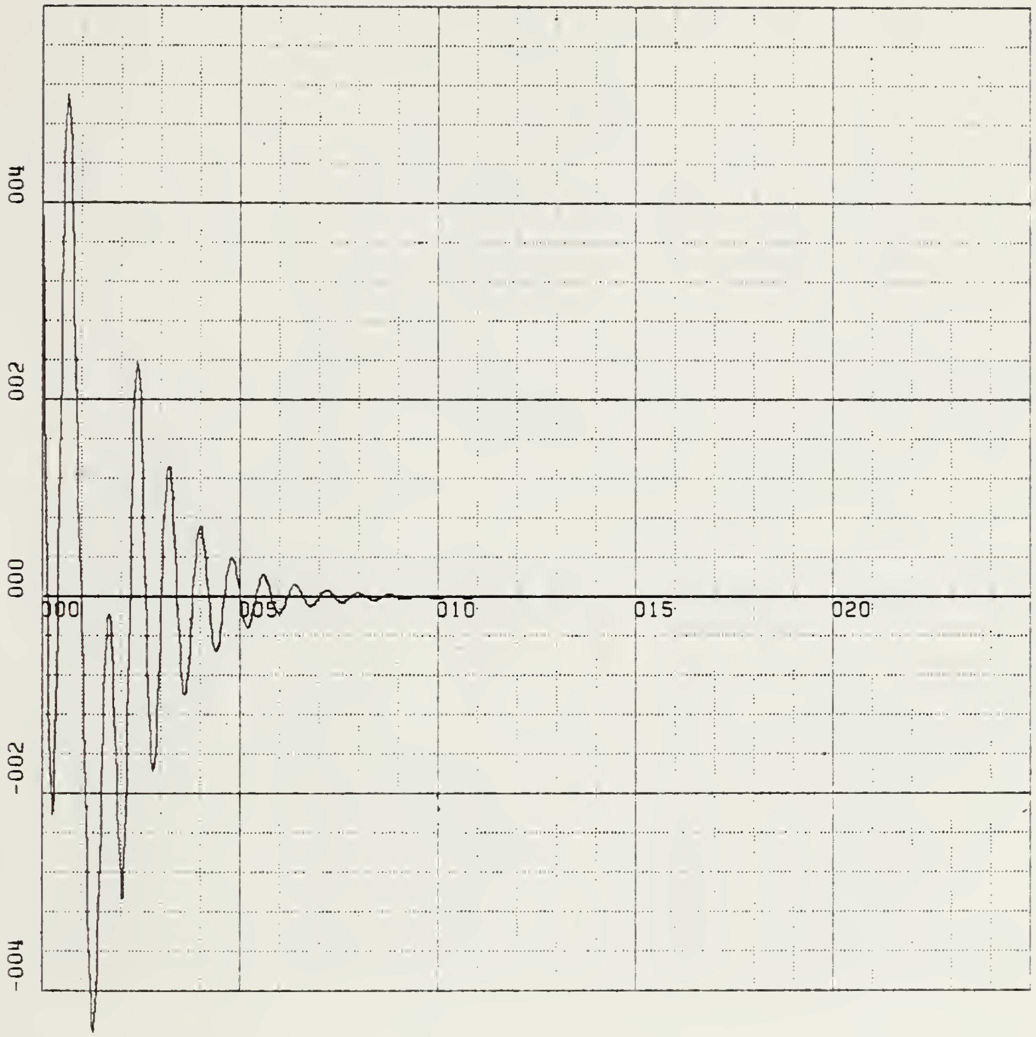
$\dot{q} = M_\upsilon^* \upsilon + M_w^* w + M_q^* q + M_\delta^* \delta$

$\dot{\theta} = q$

$\dot{h} = -w + U_0 \theta$

F. Outputs

$\upsilon$, w, q, $\theta$, h vs time of interval 0.0625
end calculation when time > 250.0



U VS TIME

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

Figure 6-2

$\upsilon$-Velocity vs Time for Specifications of Table 6-1

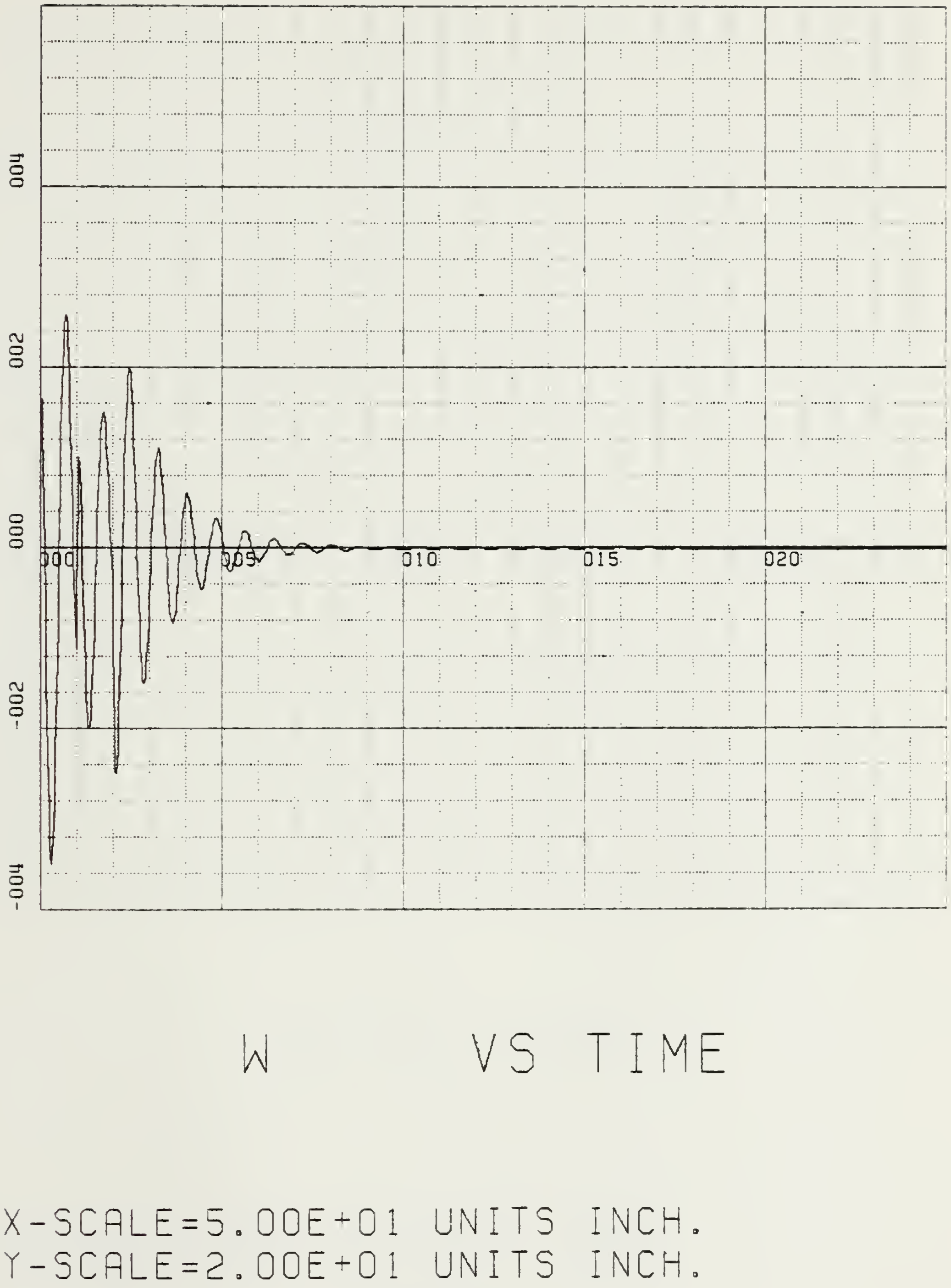
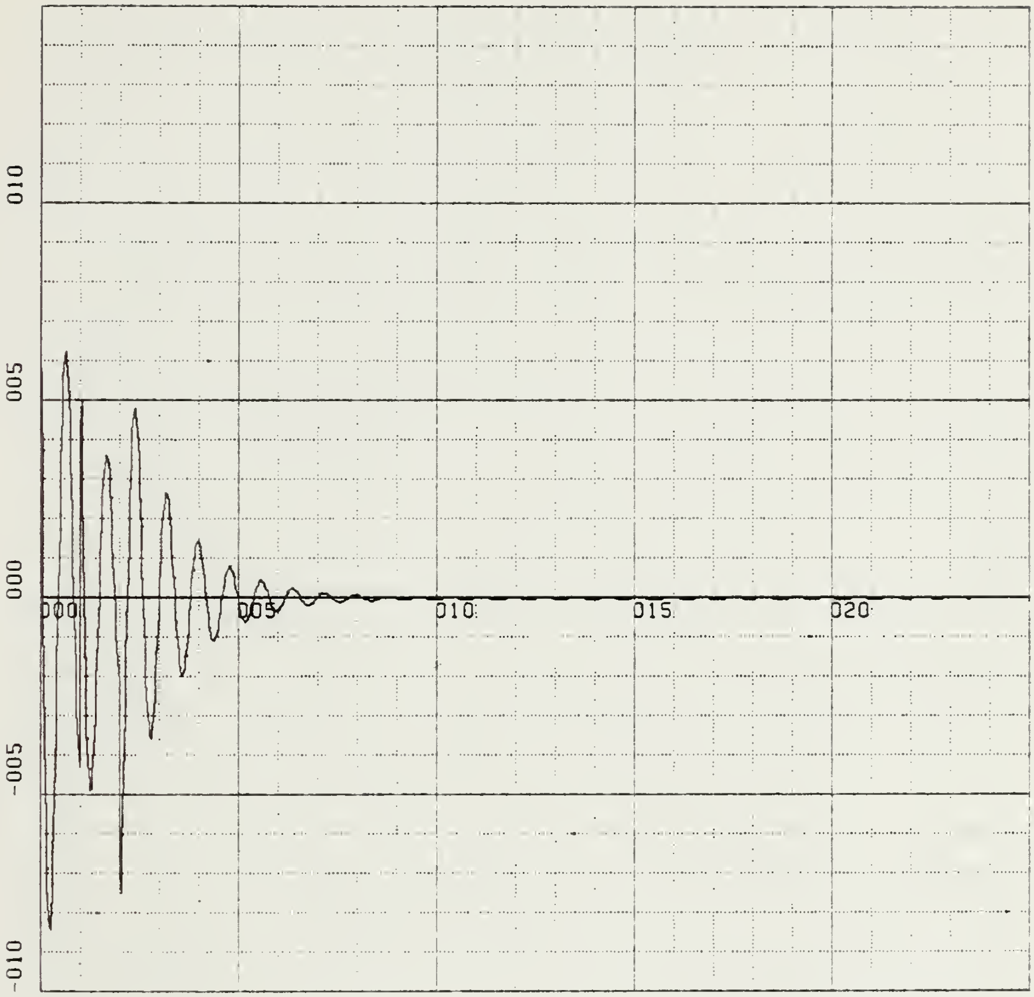


Figure 6-3

w-Velocity vs Time for Specifications of Table 6-1

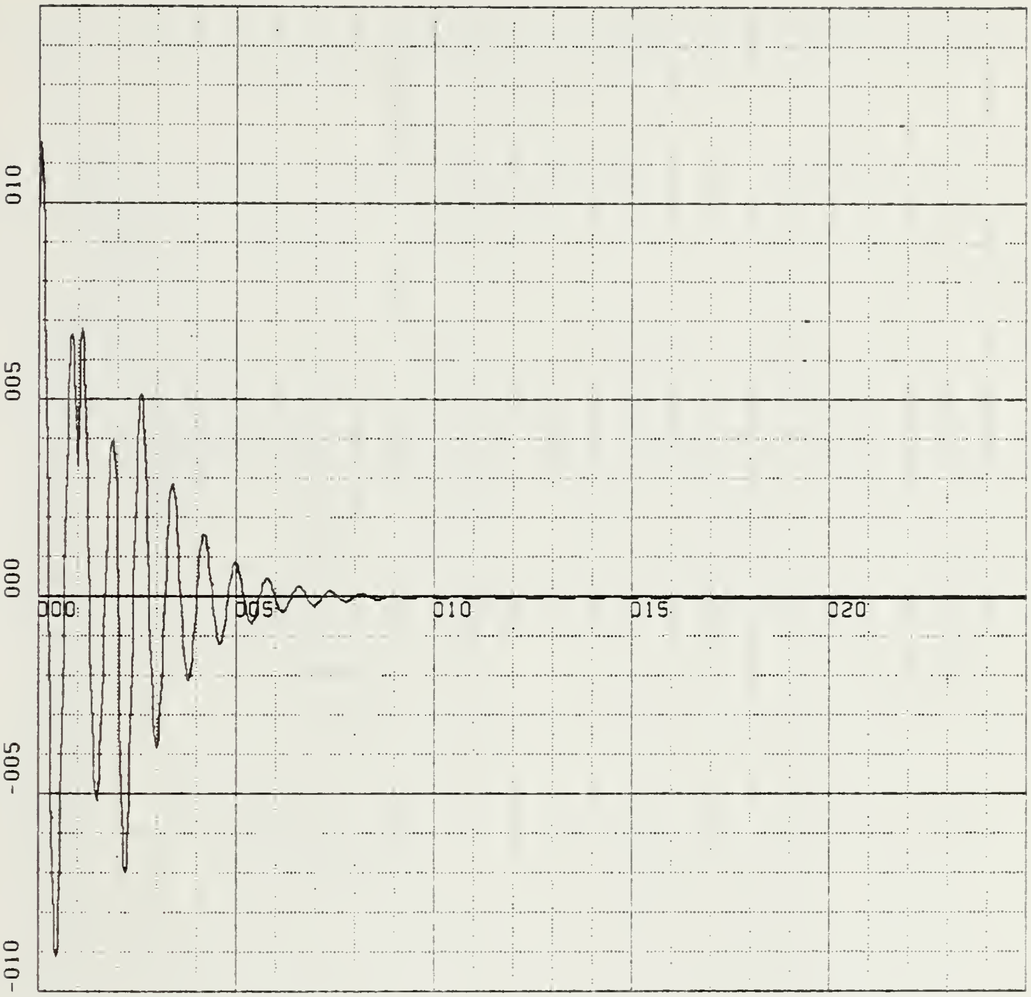


Q VS TIME

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-02 UNITS INCH.

Figure 6-4

q-Pitch Rate vs Time for Specifications of Table 6-1



THETA VS TIME

X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-02 UNITS INCH.

Figure 6-5

Θ-Pitch Angle vs Time for Specifications of Table 6-1

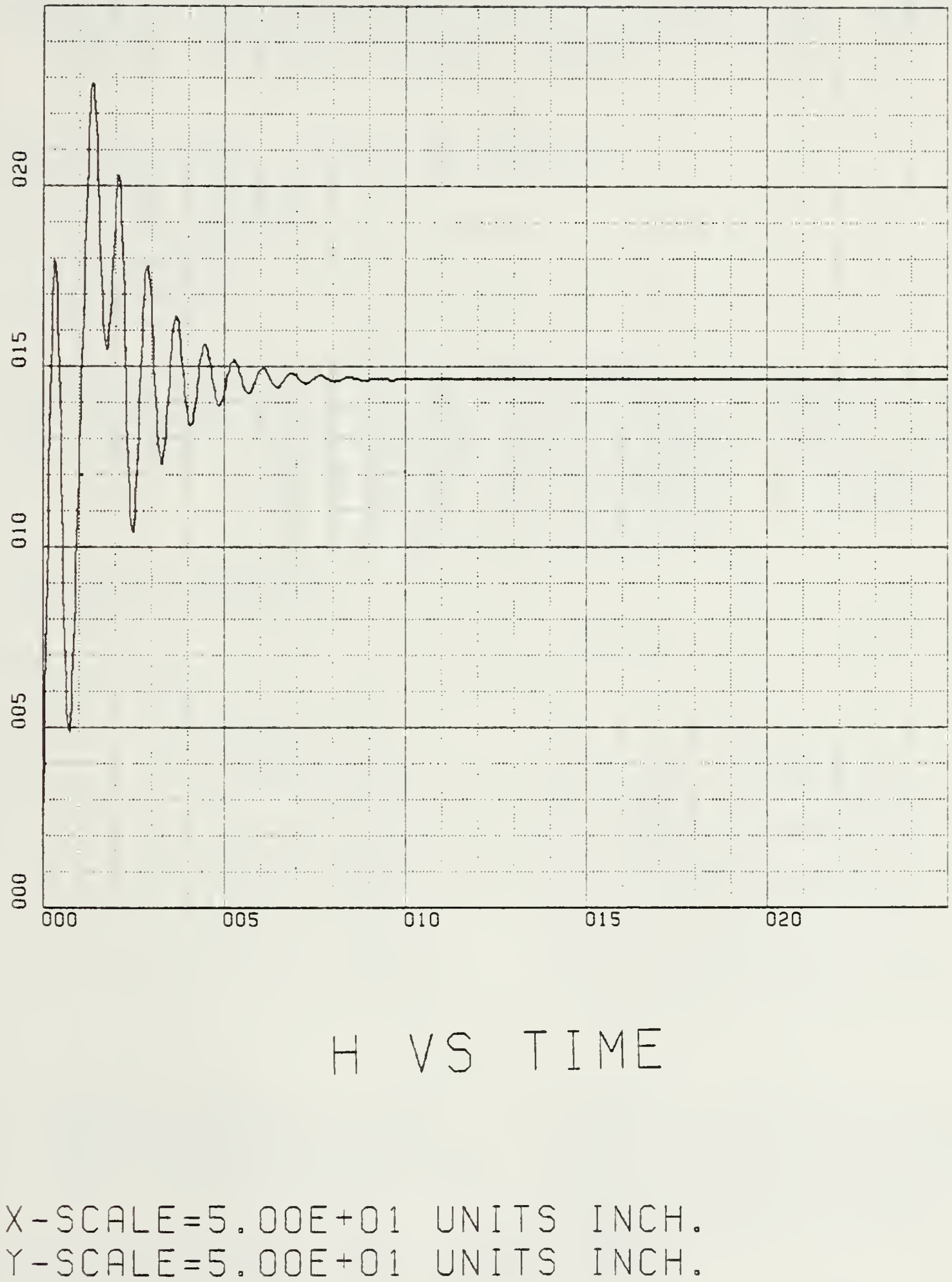


Figure 6-6

h-Height vs Time for Specifications of Table 6-1

# TABLE 6-2

## SPECIFICATIONS FOR COMPUTER SOLUTION OF THE LATERAL-DIRECTIONAL EQUATIONS DUE TO INITIAL CONDITIONS, CONTROL INPUTS AND STATE VARIABLE FEEDBACK STABILIZATION

### A. Variables and Initial Conditions

$\beta = 0.05$ rad
$p = 0.05$ rad/sec
$r = 0.075$ rad/sec
$\phi = 0.1$ rad
$\psi = 0.2$ rad
$t = 0.0$ sec

### B. Stability Derivatives and Constants

| | | | |
|---|---|---|---|
| $Y_v = -0.0829$ | $L_\beta = -4.77$ | $N_\beta = 3.55$ | $I_{xz}/I_{xx} = 0.0663$ |
| $Y_\zeta = 7.656$ | $L_p = -1.695$ | $N_p = -0.0025$ | $I_{xz}/I_{zz} = 0.0370$ |
| | $L_r = 0.1776$ | $N_r = -0.0957$ | $q = 32.174$ |
| | $L_\xi = 27.25$ | $N_\xi = -0.615$ | $U_o = 660.0$ |
| | $L_\zeta = 0.666$ | $N_\zeta = -1.383$ | |

### C. Special Functions

$$Y^*_\zeta = Y_\zeta / U_o$$

$$D = 1 - (I^2_{xz} / I_{xx} I_{zz})$$

$$L^*_\beta = [L_\beta + (I_{xz}/I_{xx}) N_\beta] / D \qquad N^*_\beta = [N_\beta + (I_{xz}/I_{zz}) L_\beta] / D$$

$$L^*_p = [L_p + (I_{xz}/I_{xx}) N_p] / D \qquad N^*_p = [N_p + (I_{xz}/I_{zz}) L_p] / D$$

$$L^*_r = [L_r + (I_{xz}/I_{xx}) N_r] / D \qquad N^*_r = [N_r + (I_{xz}/I_{zz}) L_r] / D$$

$$L^*_\xi = [L_\xi + (I_{xz}/I_{xx}) N_\xi] / D \qquad N^*_\xi = [N_\xi + (I_{xz}/I_{zz}) L_\xi] / D$$

$$L^*_\zeta = [L_\zeta + (I_{xz}/I_{xx}) N_\zeta] / D \qquad N^*_\zeta = [N_\zeta + (I_{xz}/I_{zz}) L_\zeta] / D$$

### D. Control Inputs

$\xi = 0.01 + \xi f$ for $0<t<2.5$
$\xi = -0.01 + \xi f$ for $2.5<t<5.0$
$\xi = 0.0 + \xi f$ for $t>5.0$

$\zeta = 0.0 + \zeta f$ for $0<t<15.0$
$\zeta = 0.1 + \zeta f$ for $15.0<t<17.5$
$\zeta = -0.1 + \zeta f$ for $17.5<t<20.0$
$\zeta = 0.0 + \zeta f$ for $t>20.0$

$\xi f = -0.0118\,\beta - 0.222\,p - 0.209\,r - 0.324\,\phi - 0.2667\,\psi$
$\zeta f = 0.0515\,\beta + 0.0506\,p + 0.495\,r + 0.0932\,\phi + 0.1698\,\psi$

TABLE 6-2 (Continued)

E. Derivatives

$$\dot{\beta} = Y_v \beta + r + (g/U_o)\phi + Y^*_\zeta \zeta$$

$$\dot{p} = L^*_\beta \beta + L^*_p p + L^*_r r + L^*_\xi \xi + L^*_\zeta \zeta$$

$$\dot{r} = N^*_\beta \beta + N^*_p p + N^*_r r + N^*_\xi \xi + N^*_\zeta \zeta$$

$$\dot{\phi} = p$$

$$\dot{\psi} = r$$

F. Outputs
$\beta$, p, r, $\phi$, $\psi$ vs time at interval 0.0125
end calculation when time > 50.0

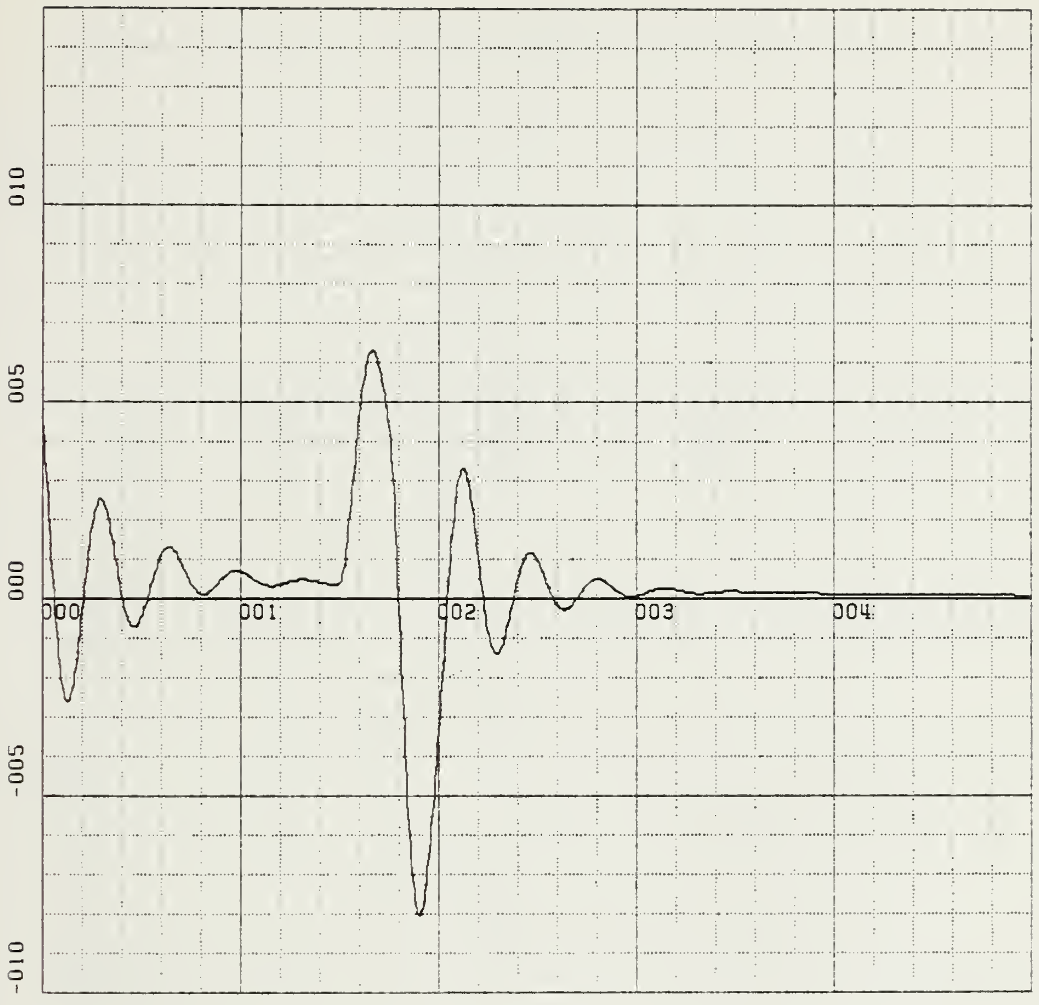
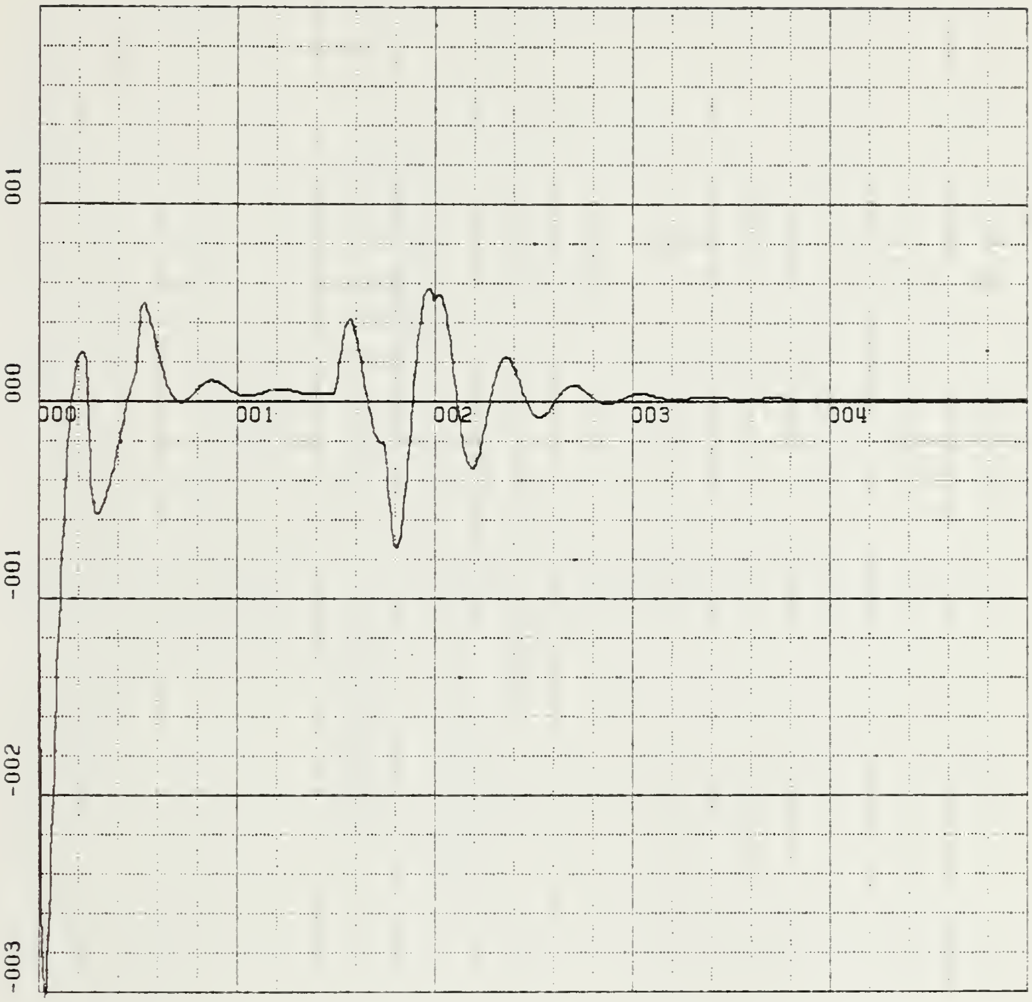


Figure 6-7

$\beta$ -Sideslip Angle vs Time for Specifications of Table 6-2

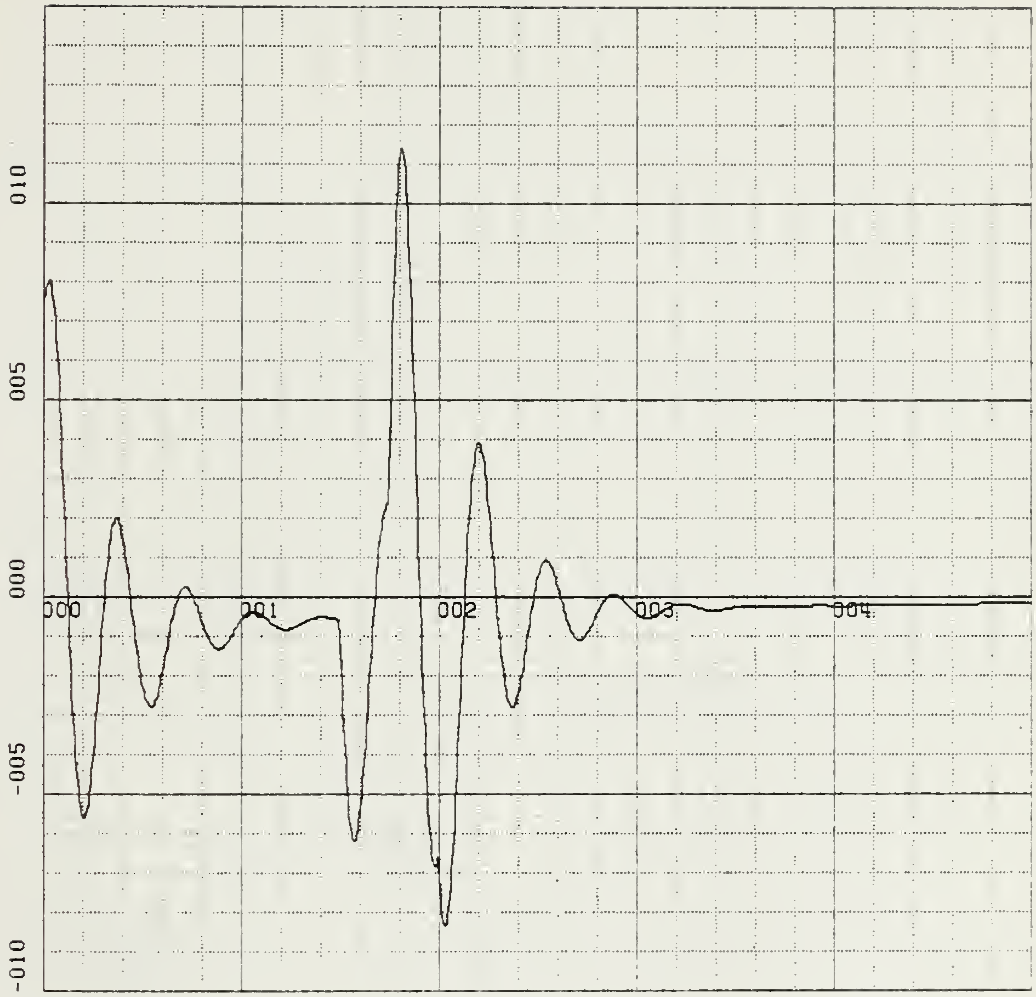


P VS TIME

X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=1.00E-01 UNITS INCH.

Figure 6-8

p-Roll Rate vs Time for Specifications of Table 6-2



R VS TIME

X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=5.00E-02 UNITS INCH.

Figure 6-9

r-Yaw Rate vs Time for Specifications of Table 6-2

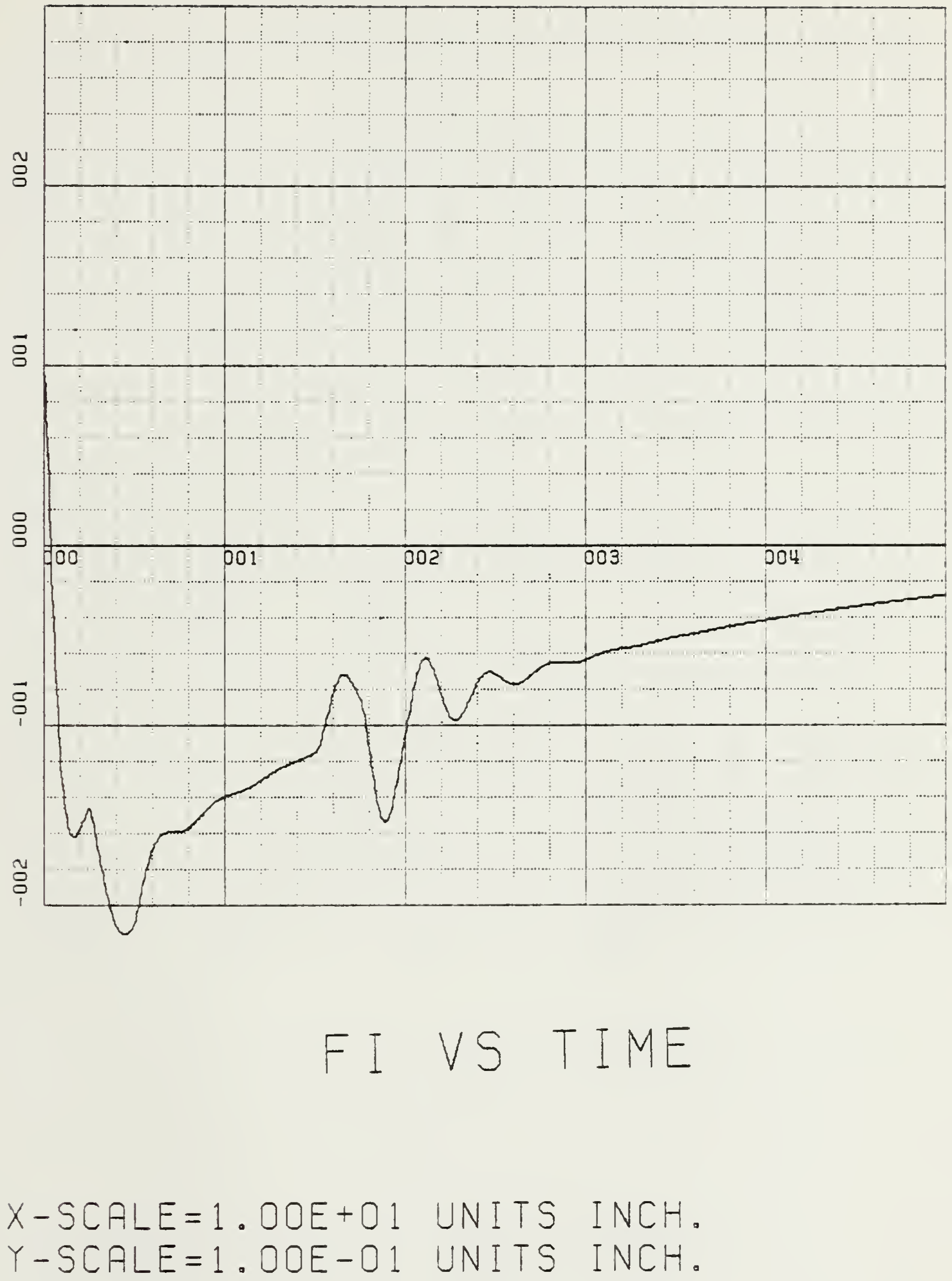
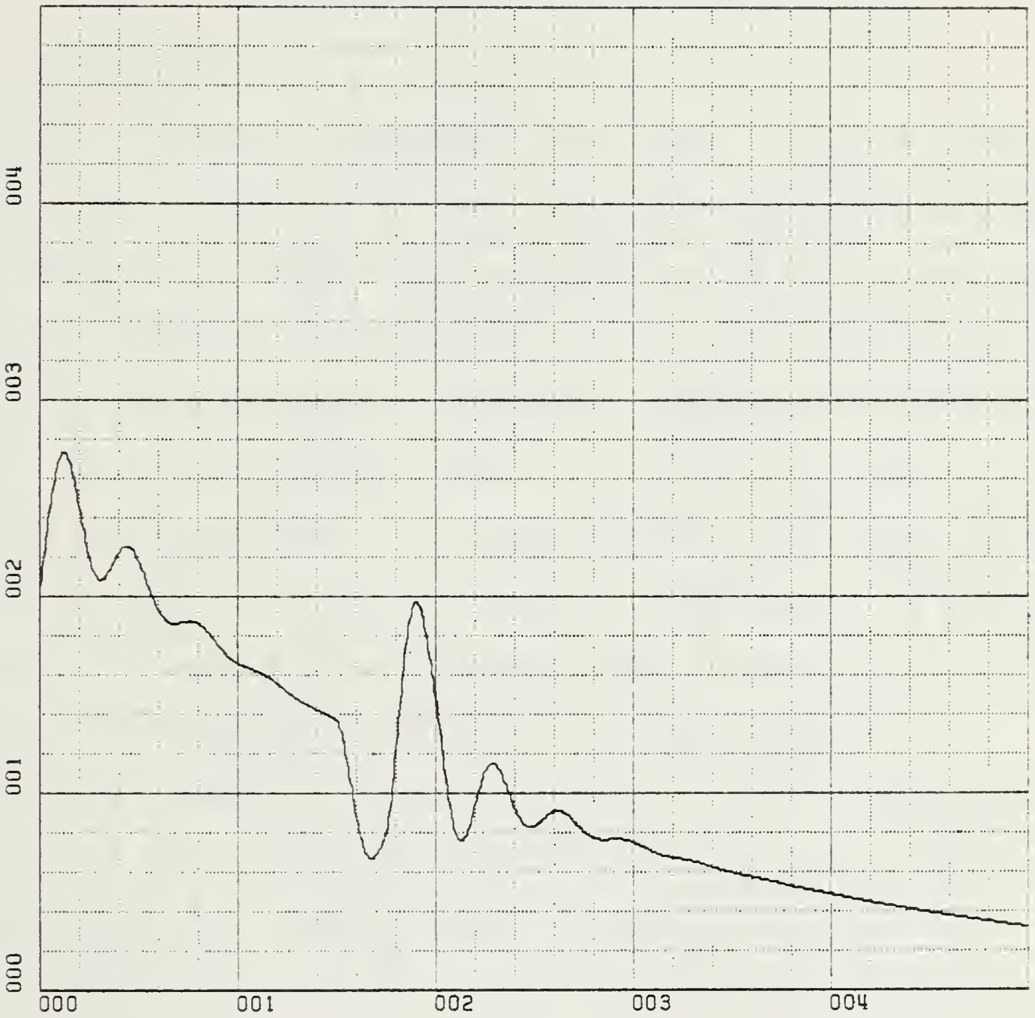


Figure 6-10

ϕ-Bank Angle vs Time for Specifications of Table 6-2



X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=1.00E-01 UNITS INCH.

Figure 6-11

$\psi$-Yaw Angle vs Time for Specifications of Table 6-2

# CHAPTER 7

# SPECIALIZATION TO MISSILES

## 7.1 INTRODUCTION

The prceeding chapters have been oriented primarily towards the dynamics of the conventional piloted airplanes. The equations of motion for the missile are still valid since it can be thought as an automatically controlled aircraft.

Missiles are found in different geometry considerations. Cruise missiles usually have no rotational symmetry, i.e. rotation through 360° is required along the missile axis to leave the external shape unchanged. Configurations with 180° rotational symmetry are seldom encountered while 120° or higher-order rotational symmetry are more typical.

In this chapter we will consider the cruciform missile, i.e., with 90° rotational symmetry to take the advantage of symmetries. This configuration is shown in Figure 7-1. This missile may be considered as typical for today's large existing missiles.

In the next sections we will consider the simplification to the equations of motion due to symmetry and the complication of concerning the coupling effect of longitudinal and lateral-directional motions. The last happens because strong rolling moments are produced when the missile is simultaneously pitched and yawed.

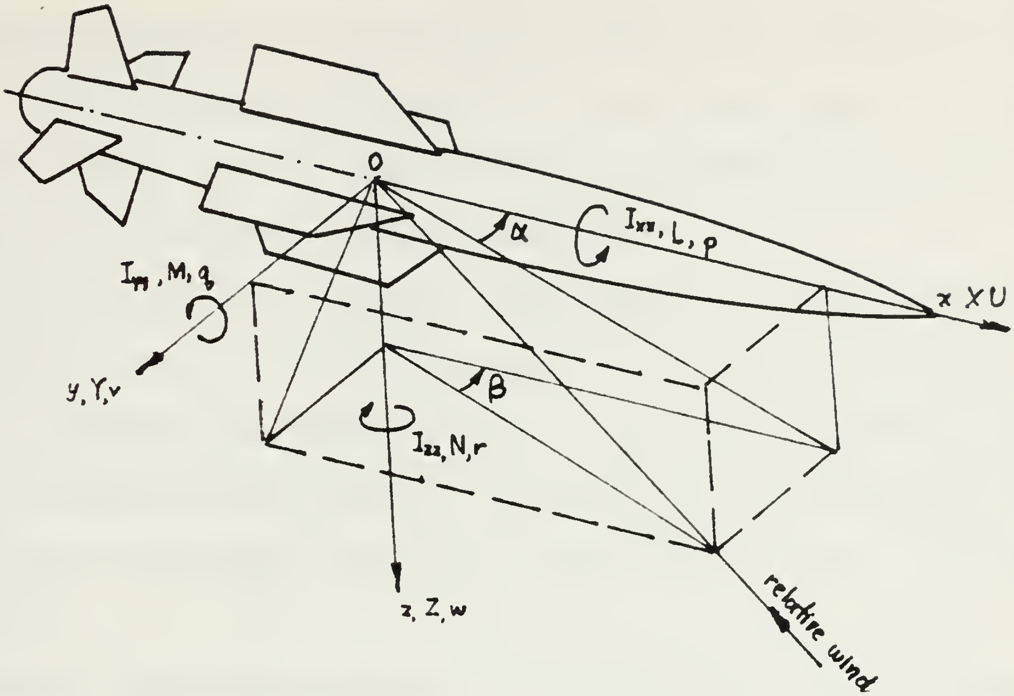


Figure 7-1

Cruciform Missile Configuration

For this particular model we will assume that control inputs are coming from the vertical and horizontal fin deflectors of the tail. A deflection of the vertical fins stands for a rudder deflection while a deflection of the horizontal fins stands for an elevator deflection . These are the only existing control inputs. Roll control is obtained by differential deflection of one vertical or horizontal fin.

## 7.2 SYMMETRY CONSIDERATIONS

For the missile's equations of motion, the reference body axes system shown in Figure 7-1 will be used as in the airplane case, and straight level horizontal flight will be assumed as the nominal flight condition.

Considering the existing symmetries, we find first that in addition to the aircraft symmetries the xy plane is a plane of symmetry and that makes the inertia tensor of the missile having only diagonal terms and as a consequence the body axes coinsides with the principal axes. Further Iyy = Izz.

The fact that the missile is not designed as an airplane, gives no tendency to remain in the same roll orientation. Vertical and horizontal find deflections can be optimized to give zero bank angle in turning maneuvers. This condition is highly desirable in missiles as well as in airplanes because the lifting forces are most efficiently generated and lift-to-drag ratio is maximum.

However, accidential actuator error signals, asymmetrical loading of the control and lifting surfaces in supersonic flight and atmospheric disturbances may introduce large roll rates and large roll rates cause cross-coupling between the longitudinal and the lateral-directional motion. This makes the equations of motion not separable in a longitudinal and a lateral-directional set which was found so convenient in the aircraft case.

We therefore find the equations of motion, Equation 2-134 and 2-135, combined in one set of equations and modified as follows:

$$\frac{1}{m}X = \dot{u} + g\theta \tag{7-1}$$

$$\frac{1}{m}Z = \dot{w} - U_0 q \tag{7-2}$$

$$\frac{1}{I_{yy}} M = \dot{q} \tag{7-3}$$

$$\frac{1}{m}Y = \dot{v} + U_0 r - g\phi \tag{7-4}$$

$$\frac{1}{I_{xx}} L = \dot{p} \tag{7-5}$$

$$\frac{1}{I_{zz}} N = \dot{r} \tag{7-6}$$

with auxilliary relations

$$p = \dot{\phi} \tag{7-7}$$

$$q = \dot{\theta} \tag{7-8}$$

$$r = \dot{\psi} \tag{7-9}$$

Before expanding the forces and moments in terms of stability derivatives we note tht due to symmetry the Y-force is the same as the Z-force which makes v=w and

$$Y_v = Z_w \tag{7-10}$$

$$Y_{\dot{v}} = Z_{\dot{w}} \tag{7-11}$$

Also the N-yawing moment is the same as the M-pitching moment which makes r=-q and

$$Y_r = -Z_q \qquad (7\text{-}12)$$

$$N_v = -M_w \qquad (7\text{-}13)$$

$$N_{\dot{v}} = -M_{\dot{w}} \qquad (7\text{-}14)$$

$$N_r = M_q \qquad (7\text{-}15)$$

because Iyy = Izz

In addition, for the control derivatives one has

$$Y_{\xi} = -Z_n \qquad (7\text{-}16)$$

$$N_{\zeta} = M_n \qquad (7\text{-}17)$$

$$N_{\dot{\xi}} = M_{\dot{n}} \qquad (7\text{-}18)$$

## 7.3 EQUATIONS OF MOTION FOR THE NON-ROLLING MISSILE

If we expand Equations 7-1 to 7-6 in terms of stability derivatives we obtain

$$\dot{u} = X_u u + X_w w - g\theta + X_n n \qquad (7\text{-}19)$$

$$\dot{w} = Z_u u + Z_w w + U_0 q + Z_n n \qquad (7\text{-}20)$$

$$\dot{q} = M_u u + M_w w + M_{\dot{w}} \dot{w} + M_q q + M_n n \qquad (7\text{-}21)$$

or substituting Equation 7-20

$$\dot{q} = (M_u + M_{\dot{w}} Z_u) u + (M_w + M_{\dot{w}} Z_w) w + (M_q + M_{\dot{w}} U_0) q + (M_n + M_{\dot{w}} Z_n) n \qquad (7\text{-}22)$$

$$\dot{v} = Y_v v - U_0 r + g\phi + Y_{\zeta} \xi \qquad (7\text{-}23)$$

$$\dot{p} = L_v v + L_p p + L_r r + \lambda_{\xi} \xi \qquad (7\text{-}24)$$

$$\dot{r} = N_v v + N_{\dot{v}} \dot{v} + N_p p + N_r r + N_{\zeta} \zeta \qquad (7\text{-}25)$$

At Equation 7-25 we would like to keep the term $N_{\dot{v}} \dot{v}$ because it is the same in magnitude as $M_{\dot{w}} \dot{w}$. Substituting Equation 7-23 for $\dot{v}$

$$\dot{r} = (N_v + N_{\dot{v}} Y_v) v + N_p p + (N_r - N_{\dot{v}} U_0) r + g N_{\dot{v}} \phi + (N_\zeta + N_{\dot{v}} Y_\zeta) \zeta \qquad (7\text{-}26)$$

If the missile is not rolling the equation is eliminated as well as the p-stability derivatives. Also, if the v-velocity perturbation is assumed zero, the equation is eliminated as well as the v-stability derivatives. Therefore, one obtains the following model for the non-rolling missile

$$\dot{w} = Z_w w + U_0 q + Z_\eta \eta \qquad (7\text{-}27)$$

$$\dot{q} = (M_w + M_{\dot{w}} Z_w) w + (M_q + M_{\dot{w}} U_0) q + (M_\eta + M_{\dot{w}} Z_\eta) \eta \qquad (7\text{-}28)$$

$$\dot{v} = Y_v v - U_0 r + Y_\zeta \zeta \qquad (7\text{-}29)$$

$$\dot{r} = (N_v + N_{\dot{v}} Y_v) v + (N_r - N_{\dot{v}} U_0) r + (N_\zeta + N_{\dot{v}} Y_\zeta) \zeta \qquad (7\text{-}30)$$

or more conveniently in matrix form

$$\begin{vmatrix} \dot{w} \\ \dot{q} \\ \dot{v} \\ \dot{r} \end{vmatrix} = \begin{vmatrix} Z_w & U_0 & 0 & 0 \\ M_w^* & M_q^* & 0 & 0 \\ 0 & 0 & Y_v & -U_0 \\ 0 & 0 & N_v^* & N_r^* \end{vmatrix} \cdot \begin{vmatrix} w \\ q \\ v \\ r \end{vmatrix} + \begin{vmatrix} Z_\eta & 0 \\ M_\eta^* & 0 \\ 0 & Y_\zeta \\ 0 & N_\zeta^* \end{vmatrix} \begin{vmatrix} \eta \\ \zeta \end{vmatrix} \qquad (7\text{-}31)$$

This model may be augmented with the auxiliary equations and transfer functions and state variable feedback can be applied for proper stabilization.

# APPENDIX A

## SOME LAPLACE TRANSFORMS

| Entry | $f(t) \quad t>0$ | $\mathcal{L}\{f(t)\} = F(s)$ |
|---|---|---|
| 1 | $\delta(t)$ | $1$ |
| 2 | $u(t)$ (unit step) | $1/s$ |
| 3 | $r(t)$ or $t$ (unit ramp) | $1/s^2$ |
| 4 | $t^2$ | $2/s^3$ |
| 5 | $e^{-at}$ | $1/(s+a)$ |
| 6 | $t e^{-at}$ | $1/(s+a)^2$ |
| 7 | $\cos \omega t$ | $s/(s^2+\omega^2)$ |
| 8 | $\sin \omega t$ | $\omega/(s^2+\omega^2)$ |
| 9 | $2 e^{-at} \cos(\omega t+\phi)$ | $e^{j\phi}/_{s+a+j\omega} + e^{-j\phi}/_{s+a-j\omega}$ |

## SOME LAPLACE TRANSFORM THEOREMS

Theorem 1: $\mathcal{L}\{a f_1(t) + b f_2(t)\} = a F_1(s) + b F_2(s)$

Theorem 2: $\mathcal{L}\{\kappa f(t)\} = \kappa F(s)$

Theorem 3: $\mathcal{L}\{df(t)/dt\} = s F(s) - f(0)$

Theorem 4: $\mathcal{L}\{d^2 f(t)/dt^2\} = s^2 F(s) - s f(0) - df/dt|_0$

Theorem 5: $\mathcal{L}\{d^3 f(t)/dt^3\} = s^3 F(s) - s^2 f(0) - s\, df/dt|_0 - d^2f/dt^2|_0$

Theorem 6: $\mathcal{L}\{\int_0^t f(\tau)d\tau\} = F(s)/s$

# REFERENCES

1. McRuer, D., Ashkenas, I. and Graham, D., Aircraft Dynamics and Automatic Control, 1st ed., Princeton University Press, 1973.

2. Etkin, B., Dynamics of Flight Stability and Control, 1st ed., John Wiley and Sons, Inc., 1959.

3. Perkins, C.D., Hage, R.E., Airplane Performance Stability and Control, 1st ed., John Wiley and Sons, Inc., 1949.

4. Dinckinson, S.L.B., Aircraft Stability and Control for Pilots and Engineers, 1st ed., Sir Isaac Pitmat and Sons LTD London, 1968.

5. Babister, A.W., Aircraft Stability and Control, 1st ed., The Macmillan Company, New York, 1961.

6. Houghton, E.L., Brock, A.E., Aerodynamics for Engineering Students, 2nd ed., St. Martin's Press, 1970.

7. NASA SP-367, Introduction to the Aerodynamics of Flight, by T.A. Talay, 1975.

8. Layton, D.M., Aircraft Dynamics Course (Class Notes), 1981.

9. Zeleny, W., Mechanics II Extended Systems Course (Class Notes), 1981.

10. Greenwood, D.T., Principles of Dynamics, 1st ed., Prentice-Hall Inc., 1965.

11. Housner, G.W., Hudson, D.E., Applied Mechanics Dynamics, 2nd ed., D. Van Nostrand Company (Canada), Ltd., 1959.

12. Carnell, P., Guided Weapon Control Systems, 2nd ed., Pergamon Press, 1980.

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